FAR-ZONE CONTRIBUTION TO UPWARD CONTINUATION OF GRAVITY ANOMALIES

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ABSTRACT

Based on a theoretical formulation of the far-zone contribution to the upward continuation of geoid-generated gravity anomalies the numerical aspects are investigated. Moreover, the numerical result over the part of the Canadian Rocky Mountains is presented in this paper.

Keywords: Dirichlet's boundary value problem, gravity, upward continuation.

1. INTRODUCTION

In order to solve the geodetic boundary value problem the gravity anomalies have to be obtained on the geoid surface. Therefore, the inverse Dirichlet's boundary value problem is solved to continue the gravity anomalies from the earth surface down onto the geoid. The downward continuation is achieved by solving Poisson's integral equation, considering that the gravity anomalies (multiplied by the geocentric radius) are harmonic at the exterior of the geoid.

In practice Poisson's integral equation is evaluated numerically. The integral equation (Fredholm's linear integral equation of the first kind (Rektorys, 1968)) is transformed into a system of linear equations that has to be computed as a whole (Martinec, 1996). However, the value of the Poisson integral kernel attenuates relatively fast for growing spherical distance, which makes the influence of gravity anomalies at larger distances from the computation point relatively small.

To reduce the number of linear equations that has to be solved, a useful approach is to divide the integration domain into the near-zone and far-zone integration sub-domains. The far-zone contribution is then subtracted from the gravity anomalies referred to the earth surface before they are downward continued solving only the near-zone contribution. Since the far-zone contribution is supposed to be much smaller and mainly a result of the variations in the anomaly field of low-frequency, it can be determined directly from an existing geopotential model from which the effect of topography is removed.

2. POISSON'S INTEGRAL

Let us begin with a definition of the geoid-generated disturbing gravity potential $T^{\mathrm{NT}}(r,\Omega)$ as a difference between the geoid-generated gravity potential $W^{\mathrm{NT}}(r,\Omega)$ and the normal gravity potential $U(r,\phi)$ given as: Vaníček et al. (2004),

$$\forall\,\Omega \in \Omega_{\mathrm{O}},\, r \in \Re^{+}:\quad T^{\mathrm{NT}}(r,\Omega) = W^{\mathrm{NT}}(r,\Omega) - U(r,\phi). \tag{1}$$

The geoid-generated gravity potential $W^{\mathrm{NT}}(r,\Omega)$ in eqn. (1) is obtained by subtracting the gravitational potential of topographical masses $V^{t}(r,\Omega)$ from the actual gravity potential $W(r,\Omega)$, so that

$$\forall\,\Omega \in \Omega_{\mathrm{O}},\, r \in \Re^{+}:\ W^{\mathrm{NT}}(r,\Omega) = W(r,\Omega) - V^{t}(r,\Omega). \tag{2}$$

The geocentric system of the orthogonal coordinates ϕ, λ and r is chosen, where ϕ and λ denote the geocentric spherical latitude and longitude $\Omega = (\phi, \lambda)$;

$\Omega \in \Omega_O$ $\left(\Omega_O \in \langle -\pi/2 \le \phi \le \pi/2; 0 \le \lambda \le 2\pi \rangle \right)$, and r is the geocentric radius $r \in \Re^+$ $\left(\Re^+ \in \langle 0, +\infty \rangle \right)$.

Since the geoid-generated disturbing gravity potential $T^{\mathrm{NT}}(r,\Omega)$ satisfies the Laplace equation at the exterior of the geoid ($\Omega \in \Omega_O \cap r > r_g(\Omega)$, where $r_g(\Omega)$ denotes the geocentric radius of the geoid surface) and is regular in infinity (Pick et al., 1973)

$$\forall\, \Omega \in \Omega_O, r > \mathrm{R}: \quad \Delta T^{\mathrm{NT}}(r,\Omega) = 0\,, \quad \lim_{r \to +\infty^+} r\, T^{\mathrm{NT}}(r,\Omega) = 0\,, \tag{3}$$

it can be expressed in terms of the solid spherical harmonics $\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$ of degree n.

According to Heiskanen and Moritz (1967, eqn. 1-87b) it reads

$$\forall\, \Omega \in \Omega_O, r \ge \mathrm{R}: \quad T^{\mathrm{NT}}(r,\Omega) = \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r} \right)^{n+1} \mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)\,. \tag{4}$$

In eqn. (4) and all the equations in the sequel, the spherical approximation of the geoid surface by the mean radius of the earth R (Bomford, 1981) is used, i.e., $\forall\, \Omega \in \Omega_O : r_g(\Omega) \approx \mathrm{R}$.

The solid spherical harmonics $\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$ in eqn. (4) are given by (e.g., Pick et al., 1973)

$$\forall\, \Omega \in \Omega_O, n \in \Im^+ \left(\Im^+ = 0, 1, \ldots, +\infty \right): \quad \mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega) = \frac{2n+1}{4\pi} \iint_{\Omega' \in \Omega_O} T^{\mathrm{NT}}(\mathrm{R},\Omega')\, \mathrm{P}_{\mathrm{n}}(\cos\psi)\, \mathrm{d}\Omega'\,, \tag{5}$$

where $\mathrm{P}_{\mathrm{n}}(\cos\psi)$ are the Legendre polynomials (Hobson, 1931) for the argument of cosine of the spherical distance ψ.

Furthermore, the fundamental formula of physical geodesy is introduced by (Heiskanen and Moritz, 1967, eqn. 2-154)

$$\forall\, \Omega \in \Omega_O, r \ge \mathrm{R}: \quad \Delta g^{\mathrm{NT}}(r,\Omega) \cong -\frac{\partial T^{\mathrm{NT}}(r,\Omega)}{\partial r} - \frac{2}{r} T^{\mathrm{NT}}(r,\Omega)\,, \tag{6}$$

where $\Delta g^{\mathrm{NT}}(r,\Omega)$ is the geoid-generated gravity anomaly.

The first term on the right-hand side of eqn. (6) stands for the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$. Regarding eqns. (4) and (6), the following relation is obtained for $\delta g^{\mathrm{NT}}(r,\Omega)$

$$\forall\, \Omega \in \Omega_O, r \ge \mathrm{R}: \quad \delta g^{\mathrm{NT}}(r,\Omega) = -\frac{\partial T^{\mathrm{NT}}(r,\Omega)}{\partial r} = \frac{1}{r} \sum_{n=0}^{\infty} (n+1) \left(\frac{\mathrm{R}}{r} \right)^{n+1} \mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)\,. \tag{7}$$

Inserting eqns. (4) and (7) into the fundamental formula of physical geodesy as described by eqn. (6), the geoid-generated gravity anomaly $\Delta g^{\mathrm{NT}}(r,\Omega)$ is expressed in the form of the solid spherical harmonics $\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$, i.e.,

$$\begin{aligned} \forall\, \Omega \in \Omega_O, r \ge \mathrm{R}: \quad \Delta g^{\mathrm{NT}}(r,\Omega) &= \frac{1}{r} \sum_{n=0}^{\infty} (n+1) \left(\frac{\mathrm{R}}{r} \right)^{n+1} \mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega) \\ &\quad - \frac{2}{r} \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r} \right)^{n+1} \mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega) \\ &= \frac{1}{r} \sum_{n=0}^{\infty} (n-1) \left(\frac{\mathrm{R}}{r} \right)^{n+1} \mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)\,. \end{aligned} \tag{8}$$

Referred to the geoid surface, eqn. (8) holds

$$\forall\, \Omega \in \Omega_O : \quad \Delta g^{\mathrm{NT}}(\mathrm{R},\Omega) = \frac{1}{\mathrm{R}} \sum_{n=0}^{\infty} (n-1)\, \mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)\,. \tag{9}$$

Summarizing the previous theory, the geoid-generated disturbing gravity potential $T^{\mathrm{NT}}(r,\Omega)$ can be described in the following form

$$\forall\, \Omega \in \Omega_O, r \ge \mathrm{R}: \quad T^{\mathrm{NT}}(r,\Omega) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\frac{\mathrm{R}}{r} \right)^{n+1} \iint_{\Omega' \in \Omega_O} T^{\mathrm{NT}}(\mathrm{R},\Omega')\, \mathrm{P}_{\mathrm{n}}(\cos\psi)\, \mathrm{d}\Omega'\,. \tag{10}$$

By analogy with eqn. (10), the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$ reads

$$\begin{aligned} \forall\, \Omega \in \Omega_O, r \ge \mathrm{R}: \quad \delta g^{\mathrm{NT}}(r,\Omega) &= \frac{1}{r} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (n+1) \left(\frac{\mathrm{R}}{r} \right)^{n+1} \\ &\quad \times \iint_{\Omega' \in \Omega_O} T^{\mathrm{NT}}(\mathrm{R},\Omega')\, \mathrm{P}_{\mathrm{n}}(\cos\psi)\, \mathrm{d}\Omega'\,. \end{aligned} \tag{11}$$

Finally, the geoid-generated gravity anomaly $\Delta g^{\mathrm{NT}}(r,\Omega)$ in eqn. (8) is given by

$$\begin{aligned} \forall\, \Omega \in \Omega_O, r \ge \mathrm{R}: \quad \Delta g^{\mathrm{NT}}(r,\Omega) &= \frac{1}{r} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (n-1) \left(\frac{\mathrm{R}}{r} \right)^{n+1} \\ &\quad \times \iint_{\Omega' \in \Omega_O} T^{\mathrm{NT}}(\mathrm{R},\Omega')\, \mathrm{P}_{\mathrm{n}}(\cos\psi)\, \mathrm{d}\Omega'\,. \end{aligned} \tag{12}$$

The Dirichlet boundary value problem, i.e., the upward continuation, is described by the Poisson integral (e.g., Kellogg, 1927; see also Bjerhamar, 1963). For the geoid-generated gravity anomaly $\Delta g^{\mathrm{NT}}[r_t(\Omega)]$ referred to the earth surface it reads

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r_t(\Omega) \geq \mathrm{R}:$$
$$\Delta g^{\mathrm{NT}}[r_t(\Omega)] = \frac{\mathrm{R}}{4\pi\, r_t(\Omega)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']\, \Delta g^{\mathrm{NT}}(\mathrm{R}, \Omega')\, \mathrm{d}\Omega', \tag{13}$$

where $\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']$ is Poisson's integral kernel.

Comparing eqn. (12) with Poisson's integral in eqn. (13), the Poisson's integral kernel is found in the following spectral form (ibid)

$$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}},\, r_t(\Omega) \geq \mathrm{R}:$$
$$\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega'] = \sum_{n=0}^{\infty} (2n+1) \left(\frac{\mathrm{R}}{r_t(\Omega)} \right)^{n+1} \mathrm{P}_{\mathrm{n}}(\cos\psi). \tag{14}$$

3. FAR-ZONE CONTRIBUTION TO UPWARD CONTINUATION

To evaluate the Poisson integral in eqn. (13), the integration domain $\Omega_{\mathrm{O}} \in \langle 0 \leq \psi \leq \pi; 0 \leq \alpha \leq 2\pi \rangle$, where α stands for the spherical azimuth, can be divided into the near-zone integration sub-domain $\Omega_{\psi_{\mathrm{o}}}$, defined on the interval $\psi \in \langle 0, \psi_{\mathrm{o}} \rangle$, and the far-zone integration sub-domain $\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}$, defined on the interval $\psi \in (\psi_{\mathrm{o}}, \pi\rangle$. The right-hand side of eqn. (13) is then rewritten as (Martinec, 1996)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r_t(\Omega) \geq \mathrm{R}:$$
$$\Delta g^{\mathrm{NT}}[r_t(\Omega)] = \frac{\mathrm{R}}{4\pi r_t(\Omega)} \iint_{\Omega' \in \Omega_{\psi_{\mathrm{o}}}} \mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\psi_{\mathrm{o}}}}\, \Delta g^{\mathrm{NT}}(\mathrm{R}, \Omega')\, \mathrm{d}\Omega'$$
$$+ \frac{\mathrm{R}}{4\pi\, r_t(\Omega)} \iint_{\Omega' \in \Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}} \mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}}\, \Delta g^{\mathrm{NT}}(\mathrm{R}, \Omega')\, \mathrm{d}\Omega', \tag{15}$$

where $\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\psi_{\mathrm{o}}}}$ and $\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}}$ are the modified Poisson's kernels for the near-zone and far-zone integration sub-domains.

As follows from the above equation, the far-zone contribution to the upward continuation is given by

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r_t(\Omega) \geq \mathrm{R}:$$
$$\Delta g^{\mathrm{NT}}_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}}[r_t(\Omega)] = \frac{\mathrm{R}}{4\pi\, r_t(\Omega)} \iint_{\Omega' \in \Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}} \mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}}$$
$$\times\, \Delta g^{\mathrm{NT}}(\mathrm{R}, \Omega')\, \mathrm{d}\Omega'. \tag{16}$$

The modified Poisson integral kernel $\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}}$ for the far-zone integration sub-domain $\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}$ is defined by (ibid)

$$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}},\, r_t(\Omega) \geq \mathrm{R}:$$
$$\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}} = \begin{cases} 0, & 0 \leq \psi \leq \psi_{\mathrm{o}}, \\ \mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega'], & \psi_{\mathrm{o}} < \psi \leq \pi. \end{cases} \tag{17}$$

According to Molodensky et al. (1960), the modified Poisson kernel $\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}}$ in eqn. (17) can further be expanded into a series of Legendre polynomials $\mathrm{P}_{\mathrm{n}}(\cos\psi)$. Thereby (Martinec, 1996)

$$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}},\, r_t(\Omega) \geq \mathrm{R}:$$
$$\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}} = \sum_{n=0}^{\infty} \frac{2n+1}{2}\, \mathrm{Q}_{\mathrm{n}}[r_t(\Omega), \psi_{\mathrm{o}}]\, \mathrm{P}_{\mathrm{n}}(\cos\psi), \tag{18}$$

where $\mathrm{Q}_{\mathrm{n}}[r_t(\Omega), \psi_{\mathrm{o}}]$ are Molodensky's truncation coefficients.

To define the Molodensky truncation coefficients $\mathrm{Q}_{\mathrm{n}}[r_t(\Omega), \psi_{\mathrm{o}}]$, eqn. (18) is multiplied by the Legendre polynomials $\mathrm{P}_{\mathrm{m}}(\cos\psi)$, so that

$$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}},\, r_t(\Omega) \geq \mathrm{R};\, n, m \in \Im^{+}:$$
$$\mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}}\, \mathrm{P}_{\mathrm{m}}(\cos\psi)$$
$$= \sum_{n=0}^{\infty} \frac{2n+1}{2}\, \mathrm{Q}_{\mathrm{n}}[r_t(\Omega), \psi_{\mathrm{o}}]\, \mathrm{P}_{\mathrm{n}}(\cos\psi)\, \mathrm{P}_{\mathrm{m}}(\cos\psi). \tag{19}$$

Furthermore, the integration with respect to the spherical distance ψ at the interval $\psi \in \langle 0, \pi \rangle$ is applied in eqn. (19)

$$\int_{\psi=0}^{\pi} \mathrm{K}[r_t(\Omega); \mathrm{R}, \Omega']_{\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}}\, \mathrm{P}_{\mathrm{m}}(\cos\psi) \sin\psi\, \mathrm{d}\psi$$
$$= \sum_{n=0}^{\infty} \frac{2n+1}{2}\, \mathrm{Q}_{\mathrm{n}}[r_t(\Omega), \psi_{\mathrm{o}}] \int_{\psi=0}^{\pi} \mathrm{P}_{\mathrm{n}}(\cos\psi)\, \mathrm{P}_{\mathrm{m}}(\cos\psi) \sin\psi\, \mathrm{d}\psi. \tag{20}$$

Using the orthogonality property of the Legendre polynomials (Hobson, 1931)

$$\forall\, n \neq m:$$
$$\int_{\psi=0}^{\pi} \mathrm{P}_{\mathrm{n}}(\cos\psi)\, \mathrm{P}_{\mathrm{m}}(\cos\psi) \sin\psi\, \mathrm{d}\psi = 0,$$
$$\forall\, n = m:$$
$$\int_{\psi=0}^{\pi} \mathrm{P}_{\mathrm{n}}(\cos\psi)\, \mathrm{P}_{\mathrm{m}}(\cos\psi) \sin\psi\, \mathrm{d}\psi$$

$$= \int_{\psi=0}^{\pi} [P_n(\cos\psi)]^2 \sin\psi \, d\psi = \frac{2}{2n+1}, \qquad (21)$$

and substituting for $K[r_t(\Omega); R, \Omega']_{\Omega_O - \Omega_{\psi_o}}$ from eqn. (18), the expression for the Molodensky truncation coefficients $Q_n[r_t(\Omega), \psi_o]$ is obtained (Martinec, 1996)

$\forall\, n = m$:

$$Q_n[r_t(\Omega), \psi_o] = \int_{\psi=\psi_O}^{\pi} K[r_t(\Omega); R, \Omega']\, P_n(\cos\psi) \sin\psi \, d\psi \,. \qquad (22)$$

The truncation coefficients $Q_n[r_t(\Omega), \psi_o]$ can be computed either by numerical integration over the interval $\psi \in (\psi_o, \pi\rangle$, or by using Paul's coefficients $R_{n,m}(\cos\psi_o)$. The Paul coefficient reads (Paul, 1973)

$\forall\, \psi \in (\psi_o, \pi\rangle, n \in \Im^+$:

$$Q_n[r_t(\Omega), \psi_o] = (2n+1)\left(\frac{R}{r_t(\Omega)}\right)^{n+1} R_{n,n}(\cos\psi_o) + \sum_{\substack{m=0 \\ m \neq n}}^{\infty} (2m+1)\left(\frac{R}{r_t(\Omega)}\right)^{m+1} R_{n,m}(\cos\psi_o) \,. \qquad (23)$$

Inserting eqn. (18) back to eqn. (16), the far-zone contribution to the upward continuation of gravity anomalies consequently becomes

$\forall\, \Omega, \Omega' \in \Omega_O, r_t(\Omega) \geq R$:

$$\Delta g^{NT}_{\Omega_O - \Omega_{\psi_o}}[r_t(\Omega)] = \frac{R}{4\pi\, r_t(\Omega)} \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n[r_t(\Omega), \psi_o] \times \iint_{\Omega' \in \Omega_O - \Omega_{\psi_O}} \Delta g^{NT}(R, \Omega')\, P_n(\cos\psi)\, d\Omega' \,. \qquad (24)$$

Regarding eqns. (5) and (9), the geoid-generated gravity anomaly on the right-hand side of eqn. (24) can be described by

$\forall\, \Omega \in \Omega_O$:

$$\Delta g^{NT}(R, \Omega) = \frac{1}{R} \sum_{n=0}^{\infty} (n-1) \frac{2n+1}{4\pi} \times \iint_{\Omega' \in \Omega_O} T^{NT}(R, \Omega')\, P_n(\cos\psi)\, d\Omega' \,. \qquad (25)$$

By applying the function $T_n^{NT}(\Omega)$ from eqn. (5) to the upward continuation of the geoid-generated gravity anomaly, the following relation for the far-zone contribution is found

$\forall\, \Omega \in \Omega_O$:

$$\Delta g^{NT}_{\Omega_O - \Omega_{\psi_o}}[r_t(\Omega)] = \frac{1}{r_t(\Omega)} \sum_{n=0}^{\infty} \frac{n-1}{2} Q_n[r_t(\Omega), \psi_o]\, T_n^{NT}(\Omega) \,. \qquad (26)$$

Taking into account also the expression of $T_n^{NT}(\Omega)$ in terms of the coefficients $C^T_{n,m}$ and $S^T_{n,m}$ (Heiskanen and Moritz, 1967)

$\forall\, \Omega \in \Omega_O$:

$$T_n^{NT}(\Omega) = \frac{GM}{R} \sum_{m=0}^{n} \left(C^T_{n,m} \cos m\lambda + S^T_{n,m} \sin m\lambda\right) P_{n,m}(\sin\phi), \qquad (27)$$

the far-zone contribution to the upward continuation of the geoid-generated gravity anomaly can be rewritten as

$\forall\, \Omega \in \Omega_O$:

$$\Delta g^{NT}_{\Omega_O - \Omega_{\psi_o}}[r_t(\Omega)] = \frac{GM}{R} \frac{1}{r_t(\Omega)} \sum_{n=0}^{\infty} \frac{n-1}{2} Q_n[r_t(\Omega), \psi_o] \times \sum_{m=0}^{n} \left(C^T_{n,m} \cos m\lambda + S^T_{n,m} \sin m\lambda\right) P_{n,m}(\sin\phi), \qquad (28)$$

where $P_{n,m}(\sin\phi)$ are the Legendre associated functions (Hobson, 1931), and GM is the geocentric gravitational constant.

4. NUMERICAL INVESTIGATION

From eqn. (28) follows that the far-zone contribution to Poisson's upward continuation is a function of the spatial distance between the computation and integration points, the degree to which the coefficients $C^T_{n,m}$ and $S^T_{n,m}$ are taken into account and the step of the numerical integration used for the computation of the truncation coefficients. To get an idea of how these dependencies are manifested in the actual value of the far-zone contribution, eqn. (28) is applied with varying heights, maximal EGM-96 retained degree and step of numerical integration. In this investigation the geodynamic coeficients $C^T_{n,m}$ and $S^T_{n,m}$ of the geopotential model EGM-96 are assumed to describe the gravity field generated by the geoid. In real, of course, they describes the earth gravity field (including the topography and the atmosphere).

The numerical integration used for the computation of the truncation coefficients $Q_n[r_t(\Omega), \psi_o]$, as defined in eqn. (22), was applied for the integration steps $\Delta\psi = 1°, 10'$ and $1'$ on the interval $\psi \in (1°, \pi\rangle$. The relative precision of the numerical integration with different step sizes is shown in Figs. 1 to 4.

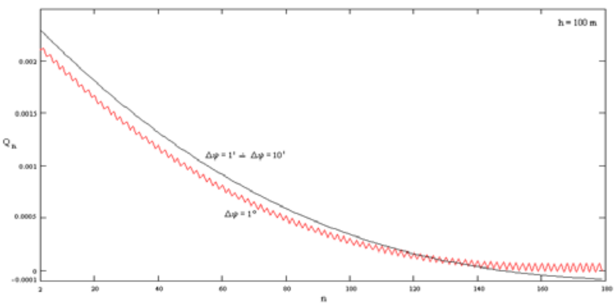

Fig. 1- Relation between the truncation coefficients $Q_n[r_t(\Omega), \psi_o]$ and the steps of numerical integration $\Delta\psi = 1^\circ, 10'$ and $1'$ for the height 100 m

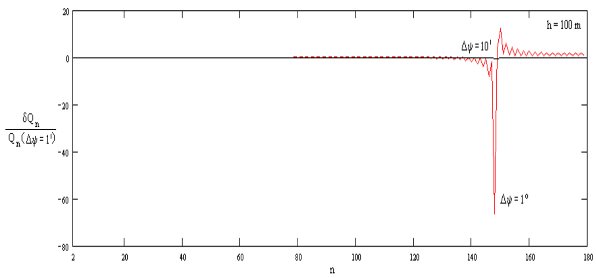

Fig. 2- The relative accuracy of the numerical integration to evaluate the truncation coefficients $Q_n[r_t(\Omega), \psi_o]$ for the height 100 m and step of numerical integration $\Delta\psi = 1^\circ$ and $10'$

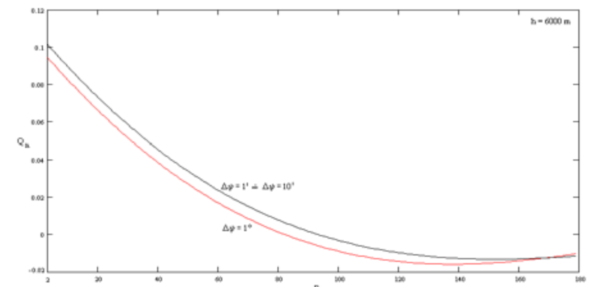

Fig. 3- Relation between the truncation coefficients $Q_n[r_t(\Omega), \psi_o]$ and the steps of numerical integration $\Delta\psi = 1^\circ, 10'$ and $1'$ for the height 6000 m

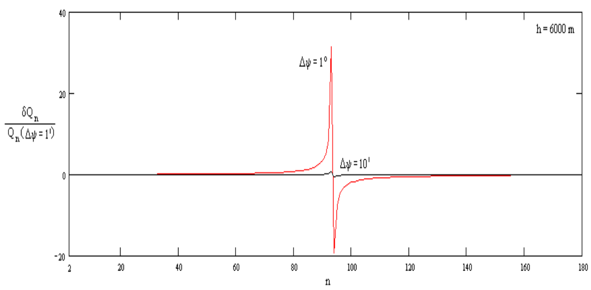

Fig. 4- The relative accuracy of the numerical integration to evaluate the truncation coefficients $Q_n[r_t(\Omega), \psi_o]$ for the height 6000 m and step of numerical integration $\Delta\psi = 1^\circ$ and $10'$

As follows from the result in Fig. 1, the far-zone contribution differs less than $5\,\mu\text{Gal}$ when 10' integration step for the numerical integration is used instead of 1'. However, when 30' step is applied the difference is only acceptable when the heights are less than 3000 m. For the heights up to 6000 m it can reach up to $40\,\mu\text{Gal}$. From these results it can be concluded that with a step of the numerical integration $\Delta\psi = 10'$ the truncation coefficients $Q_n[r_t(\Omega), \psi_o]$ can be calculated with an accuracy of about $10\,\mu\text{Gal}$.

For the calculations in this section, the global geopotential model EGM-96 is used, and the normal gravity field is defined based on parameters of the geocentric reference ellipsoid GRS-80. The harmonic part of the normal gravity field is described by the following spherical harmonics expansion (Heiskanen and Moritz, 1967)

$$\forall\, \phi \in \langle -\pi/2, \pi/2 \rangle,\; r \in \Re^+ :$$
$$V(r,\phi) = \frac{GM}{r}\left[1 + \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^{2n} C^{\text{ell}}_{2n,0}\, P_{2n,0}(\sin\phi)\right]. \qquad (29)$$

Adopting the following parameters: the major semi-axis $a = 6378137.00\ \text{m}$, the first numerical flattening $f^{-1} = 298.257$, the geocentric gravitational constant $GM = 3.986004418 \times 10^{14}\ \text{m}^3\text{s}^{-2}$ (Ries et al., 1992) and the mean angular velocity of the earth spin $\omega = 0.7292115 \times 10^{-4}\ \text{rad. s}^{-1}$ (IAG SC3 Rep., 1995), the coefficients of the series expansion in eqn. (29) are evaluated according to (ibid)

$$\forall\, n \in \Im^+ :\; C^{\text{ell}}_{2n,0} = \frac{(-1)^n}{\sqrt{4n+1}}\,\frac{3e^{2n}}{(2n+1)(2n+3)} \times \left[1 - n + \frac{5n}{e'^2}\left(\frac{e'^2 - m}{3} - \frac{2e'^2\, m}{7}\right)\right]. \qquad (30)$$

In eqn. (30), the first and second numerical excentricities e and e', and Clairaut's constant m are given by

$$e = \frac{\sqrt{a^2 - b^2}}{a}, \qquad e' = \frac{\sqrt{a^2 - b^2}}{b}, \qquad m = \frac{\omega^2 a^2 b}{GM}. \qquad (31)$$

The coefficients of GRS-80 and their corresponding EGM-96 coefficients are shown in Tab. 1.

TABLE 1: COEFFICIENTS OF THE NORMAL GRAVITY FIELD GRS-80 AND THE EARTH GRAVITY MODEL EGM-96

n	$C^{EGM96}_{n,0}$	$C^{GRS80}_{n,0}$	$C^{EGM96}_{n,0} - C^{GRS80}_{n,0}$
2	$-4.841654\cdot10^{-4}$	$-4.841679\cdot10^{-4}$	$0.025436\cdot10^{-7}$
4	$5.398739\cdot10^{-7}$	$7.903083\cdot10^{-7}$	$-2.504344\cdot10^{-7}$
6	$-1.499580\cdot10^{-7}$	$-1.687269\cdot10^{-9}$	$-1.482707\cdot10^{-7}$
8	$4.967117\cdot10^{-8}$	$3.460615\cdot10^{-12}$	$0.496677\cdot10^{-7}$

As follows from comparison of the coefficients in Tab. 1, the coefficients $C_{2,0}$ and $C_{4,0}$ are of the same order of magnitude in both series. However $C^{GRS80}_{6,0}$ is already two orders smaller than its corresponding coefficient of EGM-96. It can be shown that for a rough estimation of the maximal influence of the normalized coefficient $C^{T}_{2n,0}$ onto the gravity anomalies referred to the earth surface the following holds

$$\Delta g^{C_{2n,0}}_{\Omega_O-\Omega_{\psi_o}}\left[r_{ma}\right] < \left(\frac{GM}{2R\ r_{max}}(2n-1)\sqrt{4n+1}\,\left(1+\cos\psi_o\right) \times K\left[r_{max}\,;R,\psi_o\right]_{\Omega_O-\Omega_{\psi_o}} \right) C^{T}_{2n,0}\,. \quad (32)$$

Using the above relation it is possible to determine up to what degree the series expansion of the GRS-80 normal gravity potential field should be taken into account. Considering $r_{max} = R + 8000\,\text{m}$ and $\psi_o = 1^o$, the influence of the ellipsoidal coefficients is estimated to be

$$\left|\Delta g^{C_{2,0}}_{\Omega-\Omega_o}\right| < 492.833\,\text{mGal}\,,\ \left|\Delta g^{C_{4,0}}_{\Omega-\Omega_o}\right| < 3.238\,\text{mGal}\,,$$
$$\left|\Delta g^{C_{6,0}}_{\Omega-\Omega_o}\right| < 0.014\,\text{mGal}\,,\quad \left|\Delta g^{C_{6,0}}_{\Omega-\Omega_o}\right| < 0.001\,\text{mGal}\,.$$

The result shows that for the normal gravity potential $U(r,\phi)$ can, with an accuracy of $10\,\mu\text{Gal}$, be computed for Helmert's spheroid

$$\forall\ \phi \in \left\langle -\pi/2, \pi/2 \right\rangle, r \in \Re^{+}:$$
$$U(r,\phi) = \frac{GM}{r}\left[1 + \left(\frac{R}{r}\right)^2 C^{ell}_{2,0}P_{2,0}(\sin\phi) + \left(\frac{R}{r}\right)^4 C^{ell}_{4,0}P_{4,0}(\sin\phi)\right] + \frac{\omega^2 r^2}{3}\left(1 - P_{2,0}(\sin\phi)\right). \quad (33)$$

Consequently, the disturbing potential is given by

$$T(R,\Omega) \cong \frac{GM}{R}\sum_{n=2}^{\infty}\sum_{\substack{m=0 \\ n\neq 2,4\cap m\neq 0}}^{n}\left(C^{EGM96}_{n,m}\cos m\lambda + S^{EGM96}_{n,m}\sin m\lambda + \delta C_{4,0}\right)P_{n,m}(\sin\phi),\ \delta C_{4,0} = C^{EGM96}_{4,0} - C^{GRS80}_{4,0}. \quad (34)$$

The geopotential model EGM-96 can be used up to different degrees to calculate the far-zone contribution. Figure 5 shows for all degrees the far-zone contribution to the gravity anomaly for the heights h equal to 1000, 2000 and 6000 m. For the numerical integration the step $\Delta\psi = 10'$ is used.

Although the result for higher degrees seems to converge, it is not obvious from Fig. 5 up to which degree EGM-96 should be applied. For heights up to 1000 m an accuracy of $10\,\mu\text{Gal}$ can be achieved with the use of only 180 degrees. When the heights are larger the global geopotential model should be used up to a higher degree to reach the same accuracy. This conclusion is however only valid when we assume that the coefficients of the higher degrees are accurate, i.e. that they add information.

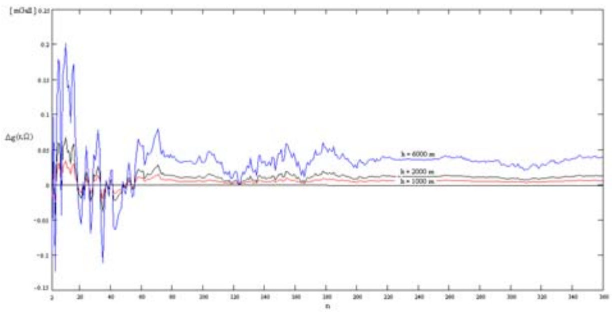

Fig. 5- The far-zone contribution to gravity anomalies on the topography computed from The EGM-96 for degrees $n = 2, 3, 4, \ldots, 360$

The far-zone contribution to the gravity anomalies referred to the earth surface at a part of the Canadian Rocky Mountains is shown in Fig. 6. At this territory it varies between –83 and 708 μGal.

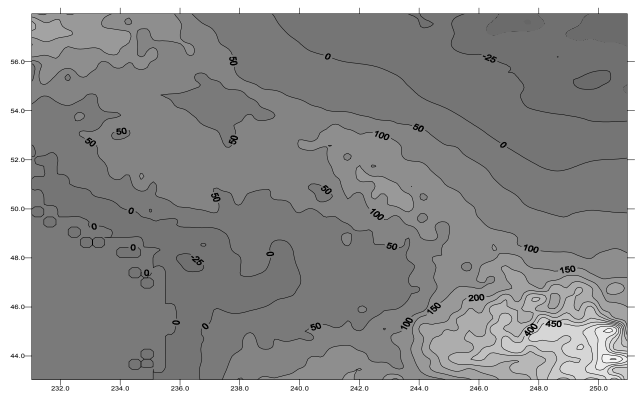

Figure. 6- The far-zone contribution to the gravity anomalies [μGal]

5. CONCLUSIONS

The far-zone contribution to gravity anomalies referred to the earth surface can directly be calculated from the spherical harmonics model for the global disturbance potential. To reach an accuracy of $10\,\mu\text{Gal}$ it is sufficient to use this model up to degree and order 180 when the heights are smaller than 1000 m.

The truncation coefficients can be calculated with sufficient accuracy using numerical integration with a 10' step.

When an ellipsoidal normal potential field is used to transform the coefficients of a global geopotential model into coefficients of the global disturbance model, the spherical harmonics expansion of the normal field can be truncated after the fourth degree.

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