CORRECTION TO HELMERT'S ORTHOMETRIC HEIGHT DUE TO ACTUAL LATERAL VARIATION OF TOPOGRAPHICAL DENSITY

Robert Tenzer
Petr Vaníček

Department of Geodesy and Geomatics Engineering, University of New Brunswick
P.O. Box 4400, Fredericton, New Brunswick, Canada, E3B 5A3; Tel.: + 1 506 458 7167
rtenzer@unb.ca

ABSTRACT

Helmert (1890) used Poincaré-Prey's gravity gradient for the definition of the orthometric height. According to this approach the gravity value needed for the evaluation of the height is obtained from the observed gravity at the earth surface reduced to the mid-point between the earth surface and the geoid, considering that the gravity gradient is constant along the plumbline. Moreover, the mean topographical density $\rho_o = 2.67\ \mathrm{g.cm}^{-3}$ is assumed to approximate the actual distribution of topographical density. The correction to Helmert's orthometric height due to the lateral variation of topographical density has been introduced by Vaníček et al. (1995). In this paper, some numerical aspects of this correction are investigated.

Keywords: Gravity gradient, orthometric height, topographical density.

1. HELMERT'S ORTHOMETRIC HEIGHT

The fundamental formula for a definition of the orthometric height $H^{O}(\Omega)$ reads (e.g., Heiskanen and Moritz, 1967, eqn. 4-21)

$$\forall\, \Omega \in \Omega_O : \qquad H^{O}(\Omega) = \frac{C[r_t(\Omega)]}{\bar{g}(\Omega)}, \tag{1}$$

where $C[r_t(\Omega)]$ is the geopotential number, and $\bar{g}(\Omega)$ is the mean value of the actual gravity along the plumbline between the physical surface of the earth and the geoid surface.

The geocentric position is given by the geocentric spherical coordinates ϕ and λ; $\Omega = (\phi, \lambda)$, and the geocentric radius r; $r \in \Re^{+}$ $\left(\Re^{+} \in \langle 0, +\infty)\right)$. In eqn. (1), $r_t(\Omega)$ further denotes the geocentric radius of the earth surface, and Ω_O stands for the total solid angle $\left[\phi \in \langle -\pi/2, \pi/2\rangle,\ \lambda \in \langle 0, 2\pi\rangle\right]$.

The geopotential number $C[r_t(\Omega)]$ is given by the difference of the actual gravity potential W_o of the geoid and the actual gravity potential $W[r_t(\Omega)]$ referred to the physical surface of the earth, so that

$$\forall\, \Omega \in \Omega_O : \qquad C[r_t(\Omega)] = W_o - W[r_t(\Omega)]. \tag{2}$$

Helmert (1890) defined the approximate value of the mean gravity $\bar{g}(\Omega)$ along the plumbline by using Poincaré-Prey's gravity gradient. It reads

$$\forall\, \Omega \in \Omega_O :$$

$$\bar{g}(\Omega) \cong g[r_t(\Omega)] - \frac{1}{2}\left.\frac{\partial g(r,\Omega)}{\partial \mathrm{t}}\right|_{r=r_t(\Omega)} H^{O}(\Omega)$$

$$\approx g[r_t(\Omega)] - \frac{1}{2}\left[\left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_{r=r_t(\Omega)} + 4\pi G \rho_o\right] H^{O}(\Omega), \tag{3}$$

where $\partial g(r,\Omega)/\partial \mathrm{t}$ represents the actual gravity gradient, and $\partial \gamma(r,\phi)/\partial \mathrm{n}$ is the normal gravity gradient.

According to this theory the gravity gradient is considered to be constant along the plumbline within the topography. Thus, the mean value of the gravity $\bar{g}(\Omega)$ is evaluated directly for the mid-point of the plumbline $H^{O}(\Omega)/2$.

From the Poisson equation (Heiskanen and Moritz, 1967, eqn. 1-14)

$$\forall\, x, y, z \in \Re \ \left(\Re \in (-\infty, +\infty)\right):$$

$$\Delta W(x,y,z) = \frac{\partial^2 W(x,y,z)}{\partial x^2} + \frac{\partial^2 W(x,y,z)}{\partial y^2} + \frac{\partial^2 W(x,y,z)}{\partial z^2}$$

$$= -4\pi G \rho_o + 2\omega^2, \tag{4}$$

and from the expression for the mean curvature of the equipotential surface $J(r,\Omega)$:

$$\forall\, \Omega \in \Omega_O\,,\, r \in \Re^+ :$$

$$J(r,\Omega) = -\frac{1}{2g(r,\Omega)}\left[\frac{\partial^2 W(x,y,z)}{\partial x^2} + \frac{\partial^2 W(x,y,z)}{\partial y^2}\right], \quad (5)$$

the Bruns formula for the actual gravity gradient can be found (Heiskanen and Moritz, 1967)

$$\forall\, \Omega \in \Omega_O\,,\, r \in \Re^+ :$$

$$\left.\frac{\partial g(r,\Omega)}{\partial \mathrm{t}}\right|_r \cong -2g(r,\Omega)\, J(r,\Omega) + 4\,\pi\, \mathrm{G}\, \rho_o - 2\omega^2\,. \quad (6)$$

In the above equations, ω denotes the mean value of the angular velocity of the earth spin, G is Newton's gravitational constant, and $\partial^2 W(x,y,z)/\partial x^2$, $\partial^2 W(x,y,z)/\partial y^2$, and $\partial^2 W(x,y,z)/\partial z^2$ are the second partial derivatives of the gravity potential in the local astronomical co-ordinate system $\mathrm{x}, \mathrm{y}, \mathrm{z}$, where the z-axis coincides with the outer normal of the local equipotential surface.

The normal gravity gradient $\partial \gamma(r,\phi)/\partial \mathrm{n}$ is defined by

$$\forall\, \Omega \in \Omega_O\,,\, r \in \Re^+ :$$

$$\left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_r = -2\gamma(r,\phi)\, J_o(\phi) - 2\omega^2\,. \quad (7)$$

Furthermore, the mean curvature of the ellipsoid surface $J_o(\phi)$ is given by (e.g., Bomford, 1971)

$$\forall\, \phi \in \langle -\pi/2, \pi/2 \rangle :\; J_o(\phi) = \frac{1}{2}\left(\frac{1}{M(\phi)} + \frac{1}{N(\phi)}\right), \quad (8)$$

where $M(\phi)$ and $N(\phi)$ are the principal radii of curvature of the ellipsoid in North-South and East-West directions.
Under the following assumption

$$\forall\, \Omega \in \Omega_O\,,\, r \in \Re^+ :$$

$$g(r,\Omega) J(r,\Omega) \cong \gamma(r,\phi) J_o(\phi)\,, \quad (9)$$

Poincaré-Prey's gravity gradient can finally be found in the form (e.g., Vaníček and Krakiwsky, 1986)

$$\forall\, \Omega \in \Omega_O\,,\, r \in \Re^+ :$$

$$\left.\frac{\partial g(r,\Omega)}{\partial \mathrm{t}}\right|_r \cong \left.\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\right|_r + 4\pi\, \mathrm{G}\, \rho_o$$
$$= -2\gamma(r,\phi)\, J_o(\phi) - 2\omega^2 + 4\,\pi\, \mathrm{G}\, \rho_o\,. \quad (10)$$

2. EFFECT OF LATERAL VARIATION OF TOPOGRAPHICAL DENSITY ON HELMERT'S ORTHOMETRIC HEIGHT

The correction $\delta H^O[\Omega : \rho(\Omega)]$ to Helmert's orthometric height due to the laterally varying topographical density $\rho(\Omega)$ is given by the following approximate expression (Vaníček et al., 1995)

$$\forall\, \Omega \in \Omega_O :$$

$$\delta H^O[\Omega : \rho(\Omega)] \approx 2\,\pi\, \mathrm{G}\left[H^O(\Omega)\right]^2 \frac{\delta\rho(\Omega)}{\gamma_o(\phi)}, \quad (11)$$

where $\delta\rho(\Omega) = \rho(\Omega) - \rho_o$ is the anomalous laterally varying topographical density, and $\gamma_o(\phi)$ is the normal gravity referred to the ellipsoid surface.

According to Martinec (1993), the laterally varying topographical density $\rho(\Omega)$ is given by

$$\forall\, \Omega \in \Omega_O :$$

$$\rho(\Omega) = \frac{1}{H^O(\Omega)} \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^O(\Omega)} \rho(r,\Omega)\, r^2\, \mathrm{d}r\,, \quad (12)$$

where $r_g(\Omega)$ denotes the geocentric radius of the geoid surface.

3. NUMERICAL INVESTIGATION

The actual lateral topographical density varies from $\rho(\Omega)_{\min} \approx 1.0\ \mathrm{g.cm^{-3}}$ (water) to $\rho(\Omega)_{\max} \approx 2.98\ \mathrm{g.cm^{-3}}$ (gabbro). Thereby, disregarding existing water bodies, the variation of topographical density $\delta\rho(\Omega)$ is within the interval $(-0.3, +0.3)$ $\mathrm{g.cm^{-3}}$ around the mean value $\rho_o = 2.67\ \mathrm{g.cm^{-3}}$. Regarding eqn. (11), it can be estimated that the variation of topographical density can cause centimetre and decimetre errors in orthometric heights, see Fig. 1.

The correction $\delta H^O[\Omega : \rho(\Omega)]$ to Helmert's orthometric height due to the lateral variation of topographical density $\delta\rho(\Omega)$ (Fig. 2) on GPS/levelling points over the territory of Canada is shown in Fig. 3. In this case the correction ranges between –1.9 cm and +3.4 cm.

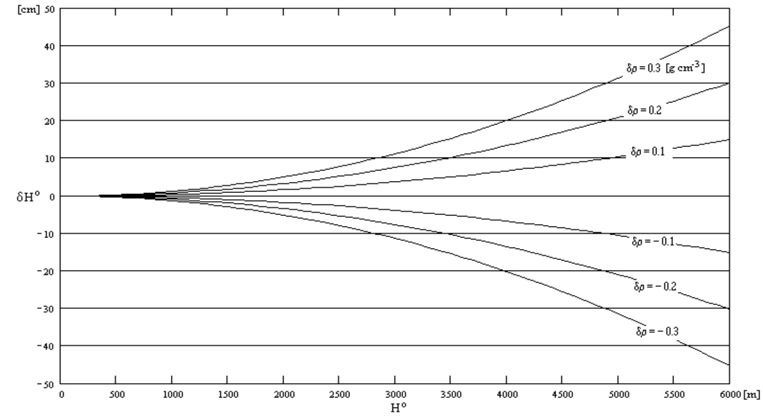


Fig. 1**-** Change of orthometric height $H^{O}(\Omega)$ due to the variation of lateral topographical density $\delta\rho(\Omega)$ [cm].

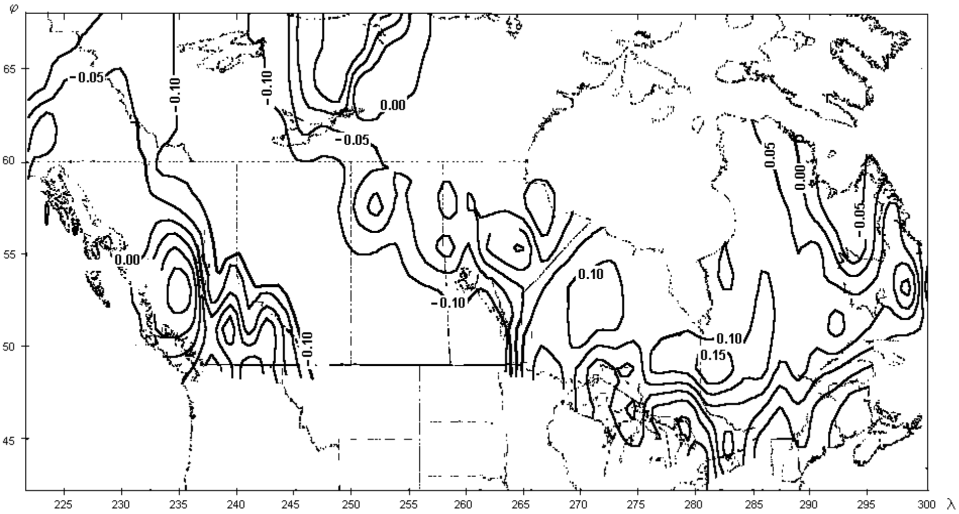
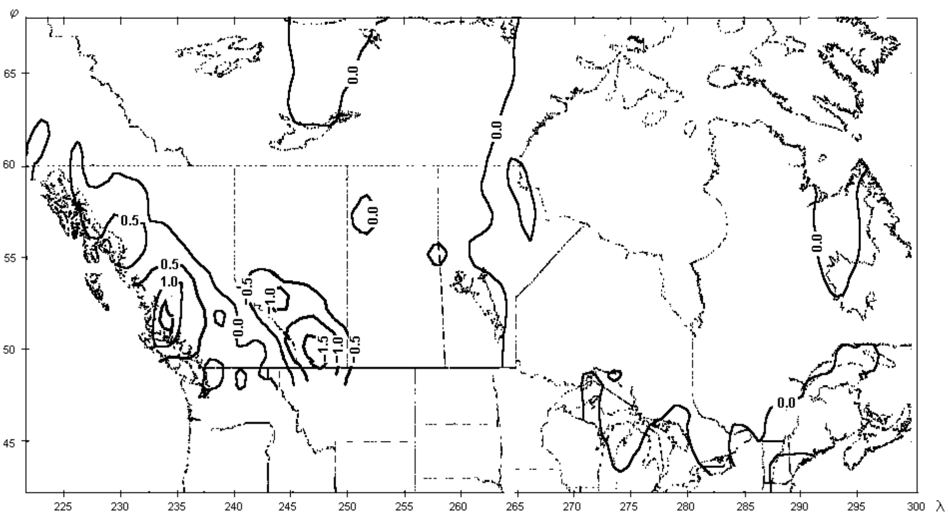


Fig. 2**-** Lateral variation of topographical density $\delta\rho(\Omega)$ at the territory of Canada [g.cm^{-3}].



Figur. 3**-** Correction to Helmert's orthometric height $\delta H^{O}[\Omega:\rho(\Omega)]$ due to the actual lateral variation of topographical density [cm].

4. CONCLUSION

As follows from the theoretical analysis in the previous chapter (see Fig. 1) the effect of the lateral variation of topographical density on the orthometric height can reach up to a several decimetres. On the other hand, the result of the numerical investigation on GPS/levelling points (where the high accuracy of the determination of the orthometric height is especially required) shows that this effect represents a change in orthometric height of only a few centimetres (see Fig. 3). It is due to the fact, that the levelling benchmarks are usually situated at locations of which height is lower than 2000 meters.

5. REFERENCES:

BOMFORD, G. **Geodesy**, 3rd edition. Clarendon Press, 1971.

HEISKANEN, W.A., MORITZ, H.,. **Physical geodesy**. W.H. Freeman and Co., San Francisco, 1967.

HELMERT, F.R. **Die Schwerkraft im Hochgebirge, insbesondere in den Tyroler Alpen. Veröff. Königl**. Preuss. Geod. Inst., No. 1, 1890.

MARTINEC, Z. **Effect of lateral density variations of topographical masses in view of improving geoid model accuracy over Canada**. Final report of contract DSS No. 23244-2-4356, Geodetic Survey of Canada, Ottawa, 1993.

VANÍČEK, P., KRAKIWSKY, E. **Geodesy**, **The concepts** (second edition). Elsevier Science B.V., Amsterdam, 1986.

VANÍČEK, P., KLEUSBERG, A., MARTINEC, Z., SUN, W., ONG, P., NAJAFI, M., VAJDA, P., HARRIE, L., TOMAŠEK, P., HORST, B. **Compilation of a precise regional geoid**. Final report on research done for the Geodetic Survey Division. Fredericton, 1995.