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Didactic transposition of irrational numbers in Portuguese textbooks from 1909 to 1963[1]

Marisa da Silva Dias[2]

ABSTRACT

The aim of this paper is to present the concept of irrational number in eleven Portuguese textbooks, spanning the period from 1909 to 1963. Under the theory of didactic transposition, with a focus on teachable knowledge, and as a bibliographic research, the main results provide definitions of irrational number as contiguous classes in textbooks in the 1910s; as a relation between incommensurable magnitudes in textbooks of the 1930s and early 1940s and as non-periodic infinite decimal fractions in textbooks of the 1950s and 1960s, as a definitive replacement of the notion of contiguous classes. In general, the didactic explanations following the concept presented in the textbooks start from a particular case of a square root of a number that is not a perfect square, along with a definition, by means of generalization, and a few examples. Examples of transcendental numbers appear after 1954.

KEYWORDS: Irrational Number. Didactic Transposition. Textbook.

Transposição didática do conceito de número irracional nas obras portuguesas de 1909 a 1963

RESUMO

O objetivo é apresentar o conceito de número irracional em onze obras didáticas portuguesas compreendendo o período de 1909 a 1963. Sob os princípios teóricos da transposição didática, com enfoque no saber a ensinar e da metodologia exploratória do tipo bibliográfica, os principais resultados voltados à publicidade desse saber são caracterizados por

[1] English version by Pedro Henrique Mattos de Oliveira. *E-mail:* dharma.pedro@gmail.com.

[2] PhD in Education. São Paulo State University (UNESP), São Paulo, Brazil. Orcid: https://orcid.org/0000-0002-4501-2625. *E-mail*: marisa.dias@unesp.br.

definições de número irracional como classes contíguas, em obras na década de 1910; como relação entre grandezas incomensuráveis, nas obras da década de 1930 e início da década de 1940 e como dízima infinita não periódica, nas obras das décadas de 1950 e 1960, em definitiva substituição a abordagem de classes contíguas. A sequência das explanações didáticas do conceito nas obras, em geral, parte de um caso particular de raiz quadrada de um número que não é quadrado perfeito, seguindo para uma definição, como generalização, e poucos exemplos. Exemplos de números transcendentes aparecem após 1954.

PALAVRAS-CHAVE: Número Irracional. Transposição Didática. Livro Didático.

Transposición didáctica del concepto de número irracional en obras portuguesas de 1909 a 1963

RESUMEN

El objetivo es presentar el concepto de número irracional en once obras didácticas portuguesas que abarcan el período de 1909 a 1963. Bajo los principios teóricos de la transposición didáctica, con un enfoque en el conocimiento a enseñar, y la metodología exploratoria bibliográfica, los principales resultados se caracterizan por definiciones de número irracional como clases contiguas, en obras de la década de 1910, como una relación entre magnitudes inconmensurables, en obras de la década de 1930 y principios de 1940 y como un diezmo infinito no periódico, en obras de las décadas de 1950 y 1960, en reemplazo definitivo del enfoque de clase contigua. La secuencia de explicaciones didácticas del concepto en las obras, en general, parte de un caso particular, la raíz cuadrada de un número que no es un cuadrado perfecto procedendo a una definición, como la generalización, y pocos ejemplos. A partir de 1954 aparecen ejemplos de irracionales trascendentes.

PALABRAS CLAVE: Número Irracional. Tranposición Didáctica. Libro de texto.

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Introduction

The interest in studying how irrational numbers are handled in textbooks arose with the research of Dias (2007) about the teaching of real numbers in Portuguese and Brazilian textbooks.

Two events motivated this study, the first was to realize that the real number only appears after 1963, considering the set of Portuguese textbooks analyzed. The second refers to conceptions of irrational numbers presented in studies by Dias (2002, 2007) and the current approach of irrational numbers in Brazilian textbooks for Basic Education.

Under analysis of production and distribution mechanisms (APPLE, 2002), textbooks can be considered as a resource that indicates knowledge involved in school education, appropriated by teachers and students. Dias (2002, 2007) brings elements that corroborate this possibility with regard to the mathematics to be taught, focusing on the concepts of irrational and real numbers.

Researchers have used textbooks, also called School Manuals, as a source of privileged information in Mathematics Education. As the various points of view converge, they provide different and promising material for theoretical deepening of mathematics teaching. Astudillo (2005) agrees that through textbooks is possible to know about "the development of a certain content, the conceptual aspects, activities, problems, exercises, the sequence and finally, the methodology that characterizes a period in history of teaching" (ASTUDILLO, 2005, p. 34, free translation).

In this paper, the analysis of the textbook focuses on a *stricto sensu* didactic transposition (CHEVALLARD, 1991), that is, to explore aspects inherent to a certain concept. About the transpositions of scientific knowledge to teachable knowledge, this study exposes the conception of authors of textbooks, who are members of the noosphere.

Chevallard (1991) calls noosphere the sphere of those who think, that is, a group of people (members) that think of the transposition. In this sense,

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the noosphere can be composed of teachers, representatives of the educational system, representatives of society, agents of public agencies, researchers in education and authors of textbooks.

In order to talk about knowledge in the didactic transposition, it is necessary to recognize the didactic system in the ternary relation between teacher, student and knowledge. Knowledge in the teacher-student pedagogical relationship cannot be scientific knowledge like thought and published by the scientific community.

Scientific knowledge must go through transformations that take into account elements inherent to the context of its dissemination to be appropriated, so that it becomes a teachable knowledge. And this one, on the other hand, occurs substantially by the teacher when transposing teachable knowledge either for the purpose of their classes, or due to teaching plans or other materials that characterize them.

This text does not intend to exhaust all aspects of an analysis, but rather to initiate a reflection on the didactic transposition of irrational numbers in the teaching of mathematics through the so-called Portuguese manuals published between 1909 and 1963.

For this purpose, what Chevallard (1991) called textualization of knowledge in the process of didacticization of knowledge was used as an analysis reference, which comprises: desyncretisation of knowledge, as a process in which the knowledge produced is divided into fields from different domains; depersonalization of knowledge, which separates knowledge from author or group of authors; programmability of knowledge acquisition, which stems from the fact that the dimension of knowledge is not addressed at once in the education system, and social control of learning, which regulates what should be taught. In addition, the transmission of knowledge comprises the publicity process to indicate what is intended to be taught about this knowledge. This paper deals more particularly with this last element, although a more in-depth analysis of all other processes is possible in the objectification of the complete textbooks.

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The methodology used in this paper is based on the principles of a bibliographic research (LIMA; MIOTO, 2007, FIORENTINI; LORENZATO, 2012). Among the procedures, the most prominent is the choice to highlight the variations in the definitions of irrational numbers and to categorize them by the approach, arranging them chronologically according to the occurrence of any new element. Thus, this article is presented and organized through the results from the definitions.

The 11 textbooks analyzed are Portuguese, selected from the private collection of José Manoel Leonardo de Matos, professor at NOVA University Lisbon. According to him, if textbooks were not the most used, they were certainly the main manuals in Portuguese schools. The textbooks date from the beginning of the 20th century, between 1909 and 1963. The seal of approval by the Ministry of National Education (Portugal) is included in all textbooks.

Irrational number as contiguous classes – publications from 1909, 1910 and 1914.

The three textbooks analyzed were written by Eduardo Ismael dos Santos Andrea, professor at Sciences College of Lisbon University and Liceu Pedro Nunes. The textbooks of 1909 and 1914 have the same title: *Complementos de álgebra: Apêndice aos Elementos de Álgebra da 3ª classe, Ensino secundário oficial 4ª e 5ª classes*. In chapter XI, entitled *Números irracionais*, consisting of four pages, there is the definition of irrational numbers related to classes.

The author began the chapter by defining, in the language of the time, a rational number:

> [...] dado um numero racional *a, qualquer* numero racional pertence necessariamente a uma das seguintes *classes*:
> 1ª Á dos numeros menores que *a*
> 2ª Á dos numeros maiores que *a*
> 3ª Á formada pelo numero *a*
> Alem d'isso, qualquer numero da primeira classe é menor que qualquer numero da segunda; na primeira classe não existe nenhum numero maior que todos os outros d'essa classe; na

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> segunda classe não existe numero que seja menor que todos os outros da classe [...]. (ANDREA, 1909, p.11 emphasis in original).

After the composition of the three classes, there is an enunciation of the order properties, as can be seen in the quotation. In the sequence, there is also a description of the property related to the density of rational numbers. The definition of irrational number does not start from the generality as it did with the rational, but with a specific number – the square root of 72 –, as follows:

> [...] é impossível obter um numero cujo quadrado seja igual a 72, de onde concluiremos que qualquer numero racional pertence necessariamente a uma das seguintes classes:
> 1ª Á classe dos numeros cujo quadrado é inferior a 72.
> 2ª Á classe dos numeros cujo quadrado é superior a 72. (ANDREA, 1909, p.12 emphasis in original).

Just as he described the properties of the disjoint classes that define a rational number, the author also enunciated them for the two formed classes that define the square root of 72 and points out that in this case there is no third class. The use of the irrational number nomenclature comes to the end: "Neste último caso diremos, *por definição*, que as duas classes consideradas determinam um numero *irracional"*. (ANDREA, 1909, p.13 emphasis in original). Following, he used the symbols of the square root of 72, $\sqrt{72}$, and briefly described equal, unequal, addition, subtraction, multiplication and division, without mention of classes structure.

Before defining an irrational number, the author states that he will name both the classes and its properties as *classes contiguas* (*contiguous classes)*, by means of abbreviation (ANDREA, 1909, p. 13).

Dias (2007) observed that the definitions of rational and irrational numbers related to classes have the same principle of cuts elaborated by Julius Wilhelm Richard Dedekind (1831-1916) in 1872. As Dedekind wrote in the preface of "Essays on the theory of numbers" (1963):

> [he] felt the need for a definition of numerical continuity, because when teaching this subject he always had to resort to geometry. Not that this didactic resource was bad, he points out, on the

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> contrary. The problem was dissatisfaction from the formal, scientific point of view. (DIAS, 2007, p. 194, free translation)

In order to seek a numerical continuity, Dedekind elaborates a form of equivalence between numbers and straight points. It is from this idea that he defines rational and irrational by cuts.

It is possible that the author of the textbook was inspired by Dedekind's ideas to place the following footnote in the chapter title:

> Neste capítulo damos breves indicações sobre a definição e calculo dos numeros irracionais, estudo que é incontestavelmente dos mais delicados para os principiantes. Embora o programa não exija explicitamente essas noções, ellas são evidentemente indispensáveis para a comprehensão nitida dos capitulos subsequentes. (ANDREA, 1909, p.11)

This explanation reinforces the relevance of the concept of irrational numbers in this textbook for the following chapters, which are: "Calculo dos Radicaes, Equação do 2º grau a uma incógnita, Theoria dos Limites, Progressões e Theoria dos logarithmos deduzida das progressões". (ANDREA, 1909)

In the textbook *Arithmetica Pratica e Geometria*, from 1910, chapter XIV – *Raiz Quadrada*, Andrea uses the same example of irrational number, $\sqrt{72}$, indicating the square root calculation method by successive approximations as well as the formation of classes:

> Vemos pois que tomando sucessivamente
> 8 , 8,4 , 8,48 , 8,485 .....
> para valor de $\sqrt{72}$, obtemos numeros cujos quadrados são todos inferiores a 72, mas esses quadrados diferem cada vez menos do numero 72 (ANDREA, 1910, p. 104).

By squaring the numbers in the quoted sequence, the author builds an idea of a class of rational numbers whose squares are less than 72, increasingly closer to that number, but never equal to it. Likewise, he presents the sequence "9, 8.5, 8.49, 8.486 ....." (ANDREA, 1910, p. 104) as a class of successively decreasing numbers whose squares are greater than 72. Thus, he defines the irrational number: "O signal $\sqrt{72}$ representa

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a existencia d'estas duas classes e chama-se numero irracional. Não é inteiro nem fraccionario" (ANDREA, 1910, p. 104).

After this demonstration, there are exercises that ask for the extraction of square roots of integers, fractional and decimal (positive) numbers[3] .

As the title suggests, the textbook is intended for the practice of Arithmetic by containing numbers, operations, divisibility, potentiation, etc. and at the end, without chapter numbering, there are practical geometry notions. There is no chapter for the irrational number – it is introduced in the chapter on square root.

It is noted that Chapter XVI – *Números complexos* does not contain imaginary numbers, as one might expect, since they were already known at the time, but as metric systems not decimals, such as with time and angle magnitudes. It is curious that the number can be complex or not complex. The number is complex when presented in the form: 3 days 8 hours 45 minutes, and not complex in a single unit composition, for example, 56 hours.

The irrational number from incommensurable magnitudes – publications from 1936 and 1940

In the textbooks by Ribeiro[4] ([1936][5]) and Tavares[6] (1940), irrational numbers appear related to incommensurable magnitudes. The *Compêndio de Álgebra e Trigonometria: para os anos 4º, 5º e 6º dos Liceus*, by Ribeiro ([1936]), has the first fourteen chapters dedicated to Algebra, andthe remaining eight to Trigonometry. Chapter V is entitled *Números irracionais* (irrational numbers) and starts with the relation of integer and fractional numbers with magnitudes: "Os números inteiros, como se sabe, apareceram pela necessidade de contar os objectos de uma colecção, ou, o que é o mesmo, de medir as grandezas descontínuas" (RIBEIRO, [1936], p. 64).

[3] The mentions of numbers in this text always refer to positive ones.
[4] Álvaro Sequeira Ribeiro was teacher at Liceu de Setúbal.
[5] The year of the textbook was obtained from the National Library of Portugal.
[6] Pedro de Campos Tavares was teacher at Liceu de Santarém.

Using fractional numbers as a new kind of number to measure continuous magnitudes is done by means of commensurable segments and measurement concept. From the following paragraph, the author introduces the concept of incommensurable magnitudes:

> Demonstra-se porém em geometria que, se quisermos medir a diagonal de um quadrado, servindo-nos do seu lado como unidade, não é possível achar uma parte alíquota do lado que caiba um número exacto de vezes na diagonal. Quere isto dizer que a medida da diagonal de um quadrado quando se toma o lado para unidade, não pode ser um número inteiro nem fraccionário, embora na prática, por aquilo que acima dissemos, se consiga sempre um fraccionário que exprima uma medida suficientemente aproximada do lado (RIBEIRO, [1936], p. 65).

After this statement, the author defines incommensurable magnitudes and the constitution of class of *rational* numbers formed by integers and fractional numbers. Consequently, to define the irrational number, the notion of contiguous classes is used, but formulated differently from the other previous textbooks (ANDREA, 1909, 1910, 1914), which are constituted only from the numerical field.

Ribeiro ([1936]) uses a drawing of two segments AB and MN, with MN smaller than AB, and assumes they are incommensurable. Considering MN as the unit of measure, the classes are formed as follows: “[a primeira] pelos números racionais que exprimem a medida de segmentos menores que AB, e outra classe constituída pelos racionais que exprimem a medida de segmentos maiores que AB” (RIBEIRO, [1936], p. 66).

This approach starts from geometry to justify the relationship between incommensurable segments and irrational numbers, without however abandoning the definition through classes of fractional numbers, which are understood as measures of commensurable segments.

The properties of contiguous classes are enunciated in the same way as in Andrea's textbook (1909, 1914), that is, pointing out not only the guarantee that any number (understood as a rational number) is in the first or second class but also the inexistence of minimum and maximum in the classes that define an irrational. It is based on the inexistence of this rational number that the author resumes the relation with the measure: "notemos ainda que *não há nenhum número racional ao mesmo tempo maior que todos os números da 1ª classe e menor que todos os da 2ª, pois se existisse, êsse número seria a própria medida de AB*" (RIBEIRO, [1936], p. 66, emphasis in original). After this argument, he defines the irrational number: "Dizemos que aquelas duas classes definem ou determinam um número de uma espécie nova, chamado irracional, que é a medida do segmento AB" (RIBEIRO, [1936], p. 67).

Next in the text, the author presents two commensurable segments and defines classes of rationals to take again the idea that both rational and irrational are defined by contiguous classes. The chapter ends by exposing the numbers called perfect squares and with a discussion about the non-existence of a square of an integer or fractional number equals 12, referring to the studies of rational arithmetic. Thus, it concludes by the formation of classes, $\sqrt{12}$ being an irrational number, that is, two classes, one formed by rational numbers whose squares are greater than 12 and the other less than 12.

The infinitude of the classes is indicated by the impossibility of determining all numbers of each class. However, to answer the need to calculate the square root of a number – which is not a perfect square – the author presents the determination of classes by the difference between numbers of each of them in the order of tenths, thousandths, etc. At the end, he concludes that "Continuando assim, podíamos obter números de uma ou outra classe cuja diferença fôsse tão pequena quanto quisessemos..." (RIBEIRO, [1936], p. 69).

The Compêndio de Álgebra e Trigonometria para os anos 4º, 5º e 6º dos Liceus (Compendium of Algebra and Trigonometry of the 4th, 5th and 6th years of Liceus), by Tavares (1940) is organized by school year, different from Ribeiro's textbook ([1936]), which is organized according to mathematical knowledge. There are sixteen chapters for Algebra, distributed in three years, and nine chapters for Trigonometry, this one destined only to the 6th year. Chapter V - Noção de número irracional

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(Notion of irrational number) is the antepenultimate of the 4th year. The first sentence of the chapter indicates that the reader already knows the commensurable and incommensurable meaning: “Sabemos da aritmética que as quantidades contínuas se classificam em comensuráveis e incomensuráveis” (TAVARES, 1940, p. 67). There is no discussion about the relation between quantities and the number field, only statements:

> Para representar a medida das quantidades comensuráveis empregam-se os números inteiros e os números fraccionários. É agora ocasião de tratar da medida das quantidades incomensuráveis.
> [...]
> Aos números que representam a medida das quantidades incomensuráveis dá-se o nome de *números irracionais* (TAVARES, 1940, p. 67).

In less than half a page, the author introduces irrational numbers with the square root example, $\sqrt{2}$, whose result is supposed to be known by reader: d = $\sqrt{2}$ l, d diagonal of a square on the l side. Initially, $\sqrt{2}$ is not treated as a number “ $\sqrt{2}$ exprime um resultado que não é igual a um número inteiro nem a um número fraccionário...” (TAVARES, 1940, p. 68). In two sentences, author states that diagonal of the square is incommensurable with its side and that $\sqrt{2}$ is an irrational number.

The text indicates that $\sqrt{2}$ is a mathematical operation contrary to potentiation, whose representation is possible only by approximation: “Não é possível representar todos os algarismos de $\sqrt{2}$, mas podemos conhecer quantos quisermos...” (TAVARES, 1940, p. 68). Using successive approximations, this statement shows two sequences of numbers, one “by default” and another “by excess” (TAVARES, 1940, p. 68), which he calls classes.

Without enunciating the properties of contiguous classes at this time, nor identifying them as such, the author develops relationships between classes and their elements strictly with the example given. Accordingly, he uses a visual aid to affirm that he separates the two classes, as follows: “0… 1 1,4 1,42 1,414… $\sqrt{2}$ … 1,415 1,42 1,5 2” (TAVARES, 1940, p. 69).

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Soon after, $\sqrt{2}$ on the arithmetic line appears as another representation, accompanied by the words:

> Representando os números racionais sôbre uma recta completa-se a correspondência entre os pontos da recta e os números, fazendo corresponder os números irracionais aos pontos da recta cuja distância à origem seja incomensurável com a distância tomada para unidade. (TAVARES, 1940, p. 69-70).

Although there is no mention of real numbers, we can see the concept of completeness that produces the equivalence elaborated by Dedekind between the points on the line and the real numbers, when introducing the irrational number.

After finishing the exploration of the irrationality of $\sqrt{2}$, a definition of contiguous classes is presented for generalization, along with all the statements that characterize these classes. The author concludes that contiguous classes define any rational number and any irrational number. The chapter ends with examples of classes that define 2/3 and $\sqrt{7}$, and a note about their infinite decimal representations with and without periodicity.

Tavares' textbook is the first of the discussed to present a representation of the irrational number on the line, although it does not present real numbers, as subsequent chapters of the 4th year cover Radicals and Powers.

The Square Root Approach – publications from 1945, 1950 and 1952

The textbooks of Tavares (1945), Ribeiro (1950) and Ribeiro (1952) are presented here in order to briefly explain how the square root is treated. Note that in previous mentioned texts square roots are thought as known to the reader. In this way, a brief explanation of the textbooks destined for the previous school years sought to observe and locate this mathematical content.

The Compêndio de Aritmética e Álgebra para os 1º, 2º e 3º anos dos Liceus (Arithmetic and Algebra Compendium for 1st, 2nd and 3rd years of

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Liceus), by Tavares, published in 1945, counts with two paragraphs before the summary entitled “Program”, and separated by subtitles “1st year” and “2nd year” – there is nothing for the 3rd year. Each paragraph contains a list of contents and none of them is about irrational numbers. The textbook follows the same organization as the textbook previously presented, divided by years. The chapter XVI - Raiz quadrada (Square root) refers to 2nd year.

At no point in the chapter the square roots of non-perfect squares are classified as irrational, the text is intended to show a method of to extract square roots from integer and fractional numbers, known in Brazil in 1980s by the key method.

The definition of square root begins with the presentation of terminology: radical, radicand and radical symbol, following the statement that it is “operação inversa da elevação ao quadrado”. (TAVARES, 1945, p. 204). The following examples refer to roots of perfect square numbers and then there is a discussion about $\sqrt{23}$ as a number between 4 and 5, in which 4 is an approximation for lack of a unit. For numbers greater than 100, a technique (key method) for extracting roots from integer and decimal numbers is presented. The exercises are directly aimed at extracting roots.

In Compêndio de Matemática (Mathematics Compendium) for 1st year of Liceus, published in 1950 and 1952, by Ribeiro, the irrational number is not mentioned. In chapter VIII from the 1950 textbook (chapter VII in 1952), Raiz quadrada (Square root), there is an initial study of square root in which a relation is made between number and the area of a square, which will be exposed next.

From three squares drawn in a checkered grid, the student is asked to use sentences to find the relation between the area to side of each square. The sentences correspond to the squares whose sides are 2 cm, 3 cm and 4 cm. Then the student is asked to find the measure of the side of a square with area of 25 cm^2 and 36cm^2, so that they can find out the regularity. In the statements of the exercises that follow, the student is asked to use the square table presented at the end of the textbook.

The next approach is with aid of a millimeter grid and board to discuss the measurement of the side of a square with area 676 mm^2. During the presentation, the square root terminology is introduced, without use of symbology. Thus, the author defines perfect square and square root of numbers by exploring square roots approximated by integers before to begin the square root extraction method. This, in turn, is presented in a similar way by Tavares (1945).

The Irrational number as a non-repeating decimal – publications from 1954, 1956 and 1963

The Compêndio de Álgebra, para 3º, 4º e 5º ano do liceu (Algebra Compendium, for the 3rd, 4th and 5th year of liceu), by Calado[7], published in 1954, begins with the Program (list of contents). The chapters are separated for each year, Chapter XVII - Noção de número irracional (Notion of irrational number) is the antepenultimate of a block of chapters for 4th year, preceded by Chapter XVI - Generalização da noção de potência (Generalization of power notion) and prior to Chapter XVIII - Radicais e potências de expoente fraccionário (Radicals and powers of fractional exponent) and Chapter XIX - Sucessões numéricas (Numerical successions).

The author begins Chapter XVII with decimal representation of fractional numbers, using an example of the fraction $\frac{3}{5}$ that "representa o *quociente exacto* da divisão de 3 por 5" (CALADO, 1954, p. 320 emphasis in original). Interestingly, the expression *exact quotient* can create the idea that 3 is divisible by 5, or, in other words, that 5 divides 3. Dias (2007) observed the presence of this conception in relation to the concept of divisibility, as well as how it reflects in the conception of irrational number. The conception that an irrational number is one that has infinite decimal representation, including repeating, was observed in the conceptions of the research subjects

[7] Professor do Liceu de Pedro Nunes.

of Dias (2002), as well as other national and international bibliographic references mentioned in the research.

The author calls the decimal representation of fractional numbers *dízima*, so dividing 3 by 5 means "*converter a fraccção em dízima*" (CALADO, 1954, p. 320, emphasis in original). Thus, it concludes that every fraction is convertible into a decimal, either limited or unlimited – in the latter case, recurring.

The notation of recurring decimals is introduced by the author with the impossibility of writing all the digits, so it is possible to: "todavia escrever tantos quantos quisermos..." (CALADO, 1954, p. 320). It is observed that the notation of infinite decimals has provided discussions that point out didactic problems capable of generating inconsistent conceptions in relation to scientific knowledge regarding the irrational number (DIAS, 2002, 2007).

The next item in the chapter is the arithmetical notion of irrational number that addresses non-periodic decimals with reference to the book of "Felix Klein - Matemática Elementar desde un punto de vista superior ,vol. I, p. 38, Madri, 1927". (CALADO, 1954, p. 321, footnote). From the possibility of "making" numbers by means of non-periodic decimals, such as "5, 1010010001..."[8] 1, exemplified by author, item ends contains a message that it is a new species number.

The subsequent item begins with the definition of an irrational number: "chama-se número irracional todo o número *decimal ilimitado e não periódico*" (CALADO, 1954, p. 322, emphasis in original), which, because it is not a repeating decimal, cannot represent a fractional number. The (only) example of an irrational number explored by the author, after defining it, is $\sqrt{2}$, not exactly as a representation of irrational number itself, but of an irrational number produced by extraction of square root, that is "... calcular o número cujo quadrado é 2, ou seja calcular $\sqrt{2}$" (CALADO, 1954, p. 322). The idea of a non-repeating, infinite decimal, is

[8] After writing the number, the author explains the logic of its construction as follows: "a seguir a cada 1, vem um grupo de zeros que tem mais um zero do que o grupo imediatamente anterior a esse 1" (CALADO, 1954, p. 321).

perceived in the calculation process of root extraction demonstrated by: "...logo reconheceremos que a operação não tem fim" (CALADO, 1954, p. 322). The author then concludes the item with: "Na prática utilizam-se as *raízes aproximadas*, ou melhor, *valores aproximados dos números irracionais*, e, por via de regra, a aproximação até às milésimas é considerada suficiente nas aplicações." (CALADO, 1954, p. 322, emphasis in original). In a footnote, the author mentions that π = 3.14159... is irrational, justifying that not all irrationals come from root extraction.

The next and last topic, still in this chapter, is The irrational number and measurement of incommensurable quantities, which starts out by introducing a segment of unidentified length and another with a measure unit of a decimeter, then proceeds to describing when two segments are commensurable or incommensurable. Then, the author mentions as an example the incommensurability of square's diagonal with its side and, returning to $\sqrt{2}$, concludes by generalizing and ending the chapter with the sentence: "... a criação dos números irracionais tornou possível, em todos os casos, a determinação da medida exacta de qualquer grandeza" (CALADO, 1954, p. 324).

In the 1956 textbook by the same author, the change is in the destination of the subject in correspondence with the Liceu year. In the 1954 textbook, the teaching of irrational numbers[9] is proposed for end of the 4th year, and in 1956 the same chapter is the second of a block of chapters destined for the 5th year, preceded by Chapter XVII - Generalização da noção de potência (Generalization of power notion).

This textbook has one less chapter, as the chapter referring to Logarithms (chapter XX found in 5th year block in the 1954 textbook) was subtracted, thus constituting twenty-two chapters. All other chapter titles and subtitles are the same in both textbooks.

[9] There is a misprint in the chapter number referring to irrational numbers in the 1956 textbook, which presents XVII instead of XVIII.

Another change concerns the order in which chapter Numeric successions is presented. In the 1954 textbook, chapter XIX is in the 1956 textbook as XVI – both are last chapters of the 4th year block, as there was a redistribution of chapters number in blocks. In the 1954 textbook, there are 9 chapters in the 3rd year block, 10 in the 4th and 4 in the 5th year. In 1956, nine chapters remain in the 3rd year block, but there are 7 in the 4th year block and 6 in the 5th year block. These changes deserve analysis with the programs of that period, which is not done here, as it is not objective of this work.

With regard to the periodic decimals explored in the textbook, it is observed that there is no mention of Stevin's work, *De thiende*, published in 1585 (BOYER, 1993, p. 232). The work approach is to represent the decimal part as an extension of the right of the unit, known today as tenths, hundredths, thousandths... This notation associated with numerical successions allowed to define irrational numbers by infinite, non-repeating, decimals. The knowledge of this relationship may have inspired authors or manual reviewers to change the order of the chapter on Numerical Successions.

The preface of Compêndio de Álgebra (Compendium of Algebra) (1º Tomo) by Silva[10] and Paulo[11], 1963, for the 6th Liceu year, indicates a review chapter

> Em particular, o capítulo I tem por objectivo principal rever e sistematizar noções adquiridas em anos anteriores; para a maioria dos alunos, bastará então ficar a ter um conhecimento nítido das propriedades operatórias que são válidas nos diversos campos numéricos e da crise que conduz ao alargamento de cada um deles; só no § 3 é introduzida matéria essencialmente nova, a qual convém dedicar atenção especial (SILVA; PAULO, 1963, PREFACE).

Unlike the other textbooks, still in preface, there is an explanation of mathematics as a historical process, contrasting with the "abstractismo inerente à matemática" (SILVA; PAULO, 1963, PREFACE).

[10] J. Sebastião e Silva full professor at Sciences Faculty of Lisbon.
[11] J. D. da Silva Paulo effective teacher at Liceu Nacional of Oeiras (Portugal).

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The textbook is composed of ten chapters. Chapter I is entitled Evolution of the number concept and has the following subdivisions: Natural numbers, Positive rational numbers, Positive numbers, Real numbers, Chapter abstract, Exercises. This is the only textbook analyzed which approaches real numbers.

The introduction of the chapter contains a half-page mentioning how numbers appear in everyday life and a historical reference saying that the concept of number spans millennia, from shepherds counting sheep to the demonstration of the irrationality of *pi*.

The item *§3. Números positivos* (*positive numbers*) begins with the following definition of irrational numbers:

> Quando, por exemplo, se procura extrair a raiz quadrada do número 2 pelo processo habitual, não se consegue nunca chegar a resto nulo nem a um período; gera-se então uma dízima infinita *não periódica*:
>
> 1, 41421356237309…([1])
>
> ([1]) Note-se que, embora infinita e não periódica, esta dízima é *bem determinada*, visto que conhecemos um *processo*, que permite calcular os seus algarismos decimais até à ordem que se queira, por maior que seja. (SILVA; PAULO, 1963, p.31, emphasis in original)

The authors define irrational number by its decimal representation – infinite non-repeating – as Calado (1954, 1956), with the difference that in the 1954 textbook the text of irrationals was aimed at 4th year students, at 5th in 1956, and, with Silva and Paulo, the definition of irrational numbers is found in textbook for the 6th year of Liceu.

The formalization of operations with irrational numbers, order and equality are based on what the author describes as "PRINCÍPIO DE CONSERVAÇÃO DE PROPRIEDADES FORMAIS" (SILVA; PAULO, 1963, p. 32, emphasis in original). The use of capital letters in the message suggests this expression is already known to reader. In addition, the item ends with a "convention" to approach the comparison between "o valor duma dízima infinita não periódica" (SILVA; PAULO, 1963, 32) and that of a finite decimal,

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also presenting the example: “$1 < \alpha < 2$; $1{,}4 < \alpha < 1{,}5$; $1{,}4142 < \alpha < 1{,}4143$;...” (SILVA; PAULO, 1963, p. 32), where α is value of $\sqrt{2}$.

The following items are about order, addition, subtraction, multiplication and division operations. The chapter ends with the topics: “approximate values” and “root extraction”. The whole approach to irrational numbers is carried out considering positive numbers; in the next paragraph, § 4 Números reais, the author introduces negative numbers and zero, to generalize that the real numbers are positive numbers, zero and negative numbers.

Conclusion

By presenting the content of the textbooks, it is possible to observe a process in which scientific knowledge becomes teachable to school environment: a textualization of knowledge. As discussed in the introduction, the textualization comprises desyncretisation, depersonalization, programmability, publicity and social control of learning. Although the emphasis here is the publicity of irrational numbers, next we point out comments and inferences because these elements are interconnected.

All works in some way mention agreement with national programs, either in the textbook's own subtitle, with a stamp or an epigraph, which indicates coherence within the noosphere between textbook, authors and agents of the Department of Education with regard to the programmability of knowledge to be taught.

In general, it is observed the new context in which irrational numbers appear in the works compared to scientific production. On the one hand, textual progression leads the adaptation of learning to the school serialization; on the other, to a sequence of knowledge. An example of this sequence is the fact that irrational numbers are presented after an explanation of potentiation. This allocation of the concept in textbooks also characterizes a desyncretization of knowledge, as not only its essence, taken here by definition, is related to other knowledge in a different way related to the production of this knowledge, but also to the very absence of elements of this same production, for example, the historical field.

The process of depersonalization of knowledge is absolute in the textbooks cited, because despite of the definition of irrational numbers tied to the notion of cuts, there is no mention of Dedekind. The same occurs in relation to the definition related to non-repeating decimals, in which, although an approach can be interpreted as close to a nested sequence of closed intervals, there is no mention of Karl Theodor Wilhelm Weierstrass (1815-1897) or even of Augustin Louis Cauchy (1789-1857) for sequences.

With regard to programmability and social control of learning, a similar process stands out in terms of didactic sequence of irrational numbers: a particular example – linked to the square root of a positive integer that is not a perfect square, a definition and another examples.

Two other examples of the programmability process can be mentioned: first is the footnote by Andrea (1909) indicating the reduction of knowledge in the explanation at the same time that it justifies the presence of this content not by the dissemination itself, but by the contents of the next chapters, and the second relates to Tavares (1940) mentioning that the meaning of commensurable and incommensurable has already been addressed at another time.

In synthesis about transmission of knowledge, it is noted in course of publications different definitions of irrational numbers. In the 1910s, the first characterization is an approach of irrational numbers as contiguous classes through the numerical field of rationals, which remains in the second characterization, along with the relation between irrational and incommensurability, in textbooks of 1936 and 1940. The third characterization is the definition of irrational through non-repeating decimals, in textbooks of 1950s and 1960s. Although there is an example of relationship between irrational numbers and incommensurable magnitudes,

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the definition by means of classes is non-existent. It can be said that in the periods and textbooks analyzed there are only two definitions of irrationals, by contiguous classes and non-repeating decimals.

From the explanations above, it is possible to observe elements of didactic transposition aimed by the authors in the textbooks on the ways to introduce and approach irrational numbers. In the 1963 textbook, it is noted that irrational numbers are no longer those that deserve a chapter, as they are already subordinate to that of real numbers, even in the absence of a conjunctive approach.

All works start out from a particular situation to define irrational numbers. The new element that appears from 1954 – Calado, 1954 and 1956, and Silva and Paulo, 1963 –is the use of transcendent numbers as examples of irrationals, but not with this terminology. It is observed that square roots are usual to introduce exemplifications of irrational numbers, even when they are not associated with geometry. In the chapters designed to explore square roots, among textbooks of Andrea (1910), Tavares (1945), Ribeiro (1950), Ribeiro (1952), Andrea's manual is the only one that makes it explicit that the example roots are irrational numbers.

From the analyzed textbooks after Ribeiro's, 1936, there are early signs of treatment of irrationals as numbers that cannot be expressed as a ratio of integers, that is, as a fractional number according to the textbooks analyzed. Such indications in Ribeiro ([1936]) appear linked to incommensurability and rational arithmetic, starting with examples of square roots. In Calado (1954), this impossibility is only mentioned in an example of transcendent number, and, in Silva and Paulo (1963), the relation between fractions and decimals is not mentioned in topic of irrational numbers, but in the preceding topic, of rational numbers, which ends with repeating infinite decimals.

A change in discourse is noted throughout of textbooks, as those that constituted the first characterization seem to be more directed to the relation between author and knowledge, whereas, gradually, in the subsequent ones, a dialogue with the reader begins. Even the number of examples has increased over the years. These elements and other aspects of didactic transposition deserve to be analyzed in detail.

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