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PAPERS

The approach to functions in mathematics textbooks before the Modern Mathematics Movement

A abordagem de funções em livros didáticos de matemática no período anterior ao Movimento da Matemática Moderna

El enfoque de las funciones en los manuales de matemáticas antes del Movimiento de la Matemática Moderna

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Abstract

The purpose of this paper is to analyze how Brazilian high school textbook authors appropriated the notion of function in the period leading up to the early 1960s, which coincides roughly with the emergence of the Modern Mathematics Movement. We will argue that this notion was appropriated by them in order to partially remedy a problem detected by Felix Klein in the teaching of mathematics in Germany: the phenomenon of Double Discontinuity. We will do this by analyzing a set of seven textbook collections that circulated in Brazil during the period in question.

Keywords: Functions; Textbook ; Modern Mathematics.

Resumo

O propósito desse trabalho é analisar de que modo os autores de livros didáticos no Brasil para o colégio se apropriaram da noção de função no período que antecede ao início da década de 1960 que coincide aproximadamente com o surgimento do Movimento da Matemática Moderna. Argumentaremos que essa noção foi por eles apropriada de forma a sanar parcialmente um problema detectado por Felix Klein no ensino da matemática na Alemanha: o fenômeno da Dupla Descontinuidade. Faremos isso através da análise de um conjunto de sete coleções de livros didáticos que circularam no Brasil no período considerado.

Palavras-chave: Funções; Livro didático; Matemática Moderna.

Resumen

El objetivo de este trabajo es analizar cómo los autores de libros de textos escolares en Brasil se apropiaron de la noción de función en el período previo a principios de la década de 1960, que coincide aproximadamente con el surgimiento del Movimiento de Matemática Moderna. Sostendremos que esta noción fue considerada por ellos para remediar parcialmente un problema detectado por Felix Klein en la enseñanza de las matemáticas en Alemania: la especificidad de la Doble Discontinuidad. Lo haremos a través del análisis de un conjunto de siete colecciones de libros de texto que circularon en Brasil durante el período considerado.

Palabras clave: Funciones; Libro de texto; Matemáticas modernas.

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Introduction

The function object has a long history, both when viewed from the perspective of the mathematical object itself and when viewed as school content. With regard to the mathematical object, its emergence can already be detected – even if unconsciously – in primitive counting processes. However, more consistent and conscious discussions about its conceptualization began to emerge mainly after the advent of Differential and Integral Calculus, with the works of Isaac Newton and Gottfried Leibniz. From the point of view of school content, discussions about its presence in the curriculum date back to the end of the 19th century and have the figure of the mathematician Felix Klein, one of its precursors. According to Ponte,

> The curricular role of the concept of function can be seen taking into account three essential aspects: (a) the more algebraic or more functional nature of the approach, (b) the generality of the concept, and (c) its application to problems and situations in real life and other sciences (PONTE, 1990, p.6).

In this sense, a problematization of the teaching of functions in Brazil ended up leading us to a historical investigation on this topic. At this point, Valente's speech inspired us:

> On the other hand, the dialogue between historical production and the present, with the day-to-day life of classrooms, cannot be relegated to a production without commitment to contemporaneity. There must be a dialogue between this production and the present. There is no way to escape this, since it is from this present that research questions arise. But this dialogue must be problematizing. A problematizing dialogue concerns the denaturalization of the elements present in the daily pedagogical practices that involve the teaching of mathematics. This is always a task for any historian: to reveal how full of historicity are elements of the present that seem to have always been the way they are (VALENTE, 2007, p.38).

When investigating the historical context of the discipline of mathematics in Brazil, we notice several reformist movements that have occurred over time. They began with the unification of the disciplines of algebra, arithmetic and geometry, which had previously been treated independently. This process resulted in the creation of the discipline of mathematics in 1929, with Professor Euclides Roxo as one of its main advocates. In addition to this merger, Roxo also advocated the inclusion of the concept of function in secondary education. In the more general teaching plan, the *Francisco Campos* (1931), *Capanema* (1942) and *Simões Filho* (1951) Reforms took place between the 1920s and 1950s. From the 1960s onwards, the Modern Mathematics Movement (MMM) emerged in Brazil, with Professor Osvaldo Sangiorgi as one of its central figures. This movement sought a different vision for teaching Mathematics, both in the way the content was transmitted and in the content itself. In this process, the theme of Functions began to coexist both at the beginning of secondary education and in what was conventionally called high school.

Some studies, such as Braga (2003) and Oliveira (2009), analyzed how the idea of function appeared in textbooks during these movements, in the first cycle of secondary education. Braga (2003) analyzed a set of textbooks from the period, while Oliveira (2009) compared two collections. Our objective in this study is to analyze how authors of textbooks for high schools in Brazil appropriated the notion of function in the period before the beginning of the 1960s, that is, before the MMM, taking a set of 07 (seven) collections that were published at that time.

Theoretical framework

The Textbook (TB) has been frequently used in various works on the History of Education and in a very diverse way, depending on the problems addressed by its authors. An object that is difficult to conceptualize, but endowed with specificities that make it unique within the school environment, it plays a singular role when taken as a historical source. One of the pioneering works in Brazil in the field of History of Education that mobilizes this source is that of Circe Bittencourt. According to this author,

> Textbooks are a controversial subject, as they generate radical positions among teachers, students and researchers of educational problems. The main consumers of textbooks, teachers and students, differ in their assessment of the role they play in school life. For some teachers, textbooks are considered an obstacle to learning, a work tool to be discarded in the classroom. For others, they are fundamental material to which the course is totally subordinated. In practice, textbooks have been used by teachers, regardless of their use in the classroom, to prepare 'their classes' at all levels of schooling, whether to plan the school year or to systematize the academic content.
>
> textbooks, or simply as a reference in the preparation of exercises or questionnaires (BITTENCOURT, 2008, p.13).

With such a large number of situations that qualify it, it is no wonder that TB receive so much attention from researchers in the most diverse fields, particularly in the History of Education. In his text "The historian and the textbook", Alain Chopin lists several considerations regarding this object, which we will use to reinforce our commitment to using it as a source. Initially, the author mentions the symbolic characteristic that TB have, when he compares them to other symbols of national culture. According to Chopin,

> The manual is effectively inscribed in material reality, participates in the cultural universe and stands out, in the same way as the flag or the currency, in the sphere of the symbolic. As a repository of educational content, the manual has, above all, the role of transmitting to the younger generations the knowledge and skills (even the "know-how") which, in a given area and at a given time, are considered indispensable for society to perpetuate itself (CHOPIN, 2002, p. 14).

In addition, as expected, the TB is mentioned by Chopin in the context of its function as a carrier of the content to be taught. Going further, the author also identifies another function that the textbook has, which is that of conveying value systems that end up instilling in students an ideology, a system of beliefs, standing out as a participant in the student's socialization process (CHOPIN, 2002, p. 14), as well as a pedagogical instrument, "[...] insofar as it proposes learning methods and techniques, which official instructions or prefaces could only provide as objectives or guiding principles." (CHOPIN, 2002, p.14). Since we are particularly interested in a period, its use as a source seemed appropriate to us, since,

> The Manuals are therefore particularly suited to serial study. By focusing on manuals, historians can thus observe, over the long term, the emergence and transformations of a scientific notion, the inflections of a pedagogical method or the representations of social behavior (CHOPIN, 2002, p.15).

Because, at times, we deal with more than one edition, we feel comfortable in being able to find support for this argument, since,

> The manual is part of continuity: except in the case where a subject is removed from the programs, the production of manuals is never exhausted: new works replace editions deemed obsolete - this is the rule in countries that have a state edition - or establish competition with older products, which are abundantly reissued. But these reissues are not justified only by the renewal of new generations and the material wear and tear of the works: reissues do not necessarily lead to repetition. The similarity of the titles does not necessarily conceal identical content and the modifications made to the text or iconography do not occur only when the program changes. (CHOPIN, 2002, p.15).

In this way, these elements pointed out by Chopin gave us a solid foundation from which we could follow the path of analyzing textbooks.

A particularly useful concept for our analyses and which will appear frequently in this work is that of appropriation, which emerged mainly from the works of the French historian Roger Chartier. Present in discussions on Cultural History, this term, along with another, that of representation, provides an important theoretical contribution to research in the History of Mathematics Education. In the specific case of appropriation, we let the author himself speak about his perspective:

> A concept that seems useful to us here is that of appropriation, because, understood in more sociological than phenomenological terms, the notion of appropriation makes it possible to evaluate the differences in cultural sharing, in the creative invention that lies at the heart of the reception process (CHARTIER, 1992, p. 232-233).

Thus, and this is very important here, the mechanism of appropriation, being subsequent to reception, makes it possible to understand how the information that textbook authors receive during their formative process or even at the time of writing a text, is converted into meaning. Biccas (2012, p. 286) summarizes by saying that it is "[...] the construction of meanings, of ways of interpreting." In our specific case, this perspective fits well, since textbook authors are also readers and, as Biccas (2012, p. 287) points out, it is necessary to "[...] recover the reader and their context, in order to seek the way in which they appropriated the texts." In addition, the manifestation of the ways in which the authors produced their discourses can be revealed through the analysis of textbooks.

Some methodological procedures

In order to achieve our objectives, it was necessary to locate books published in Brazil during the period considered. We used Oliveira Filho's thesis (2013) in two ways: it served as a guide for the periods to be investigated and enabled us to identify most of the books that were being produced and circulated during the period studied. In fact, the author presents an evolution of the structuring of mathematics teaching in schools through a division into phases. When identifying these phases, he lists mathematics books found and that were published in each of them. It is important to note that, due to the difficulty in locating some books, the edition we obtained may not have been published in the phase in which Oliveira Filho (2013) placed it. In Chart 1, we present the phases and the books he found.

Chart 1 – Phases and Books found in the period before 1960

Fase	Some books found by Oliveira Filho (2013)
Cursos Complementares (1931-1942)	• *Lições de matemática – Thales e Mello Carvalho – 1938* • *Pontos de matemática – Gumercindo Lima – 1938* • *Lições de matemática para médicos e químicos – Alberto Serrão – 1941*
The creation of the 2nd Cycle of the Secondary Course and the consolidation of the school subject Mathematics at the College (1942-1951)	• *Matemática 2.º Ciclo – 3.ª Série – 3.ª edição – 1949 – Livraria Francisco Alves.* • *Matemática para os Cursos Clássico e Científico – 3.ª Série – Thales Mello Carvalho – 2.ª edição – 1948 – Companhia Editora Nacional.* • *Curso de Matemática – 3.º Livro – Ciclo Colegial – Algacyr Munhoz Maeder – 2.ª edição – 1949 – Edições Melhoramentos.*
Stabilization of high school mathematics (1951 onwards)	• "*Matemática 2.º Ciclo – 1.ª Série – 2.ª Série – 3.ª Série Euclides Roxo, Roberto Peixoto, Haroldo Cunha, Dacorso Netto".* • *Coleção "Matemática para os Cursos Clássico e Científico – 1.º Ano, Colegial – 2.º Ano Colegial – 3.º Ano Colegial – Thales Mello Carvalho – Companhia Editora Nacional".* • *Coleção "Curso de Matemática – 1.º Livro Ciclo Colegial – 2.º Livro Ciclo Colegial – 3.º Livro Ciclo Colegial – Algacyr Munhoz Maeder – Edições Melhoramentos".* • *Coleção "Matemática – Para o Primeiro Ano Colegial – Para o Segundo Ano Colegial – Para o Terceiro Ano Colegial – Ary Quintella – Companhia Editora Nacional".* • *Coleção "Matemática" – 1ª. Série – Curso Clássico e Científico – 2ª. Série Curso Clássico e Científico – 3ª. Série – Curso Clássico e Científico – Edison Farah, Omar Catunda, João Batista Castanho e Benedito Castrucci.*[1]

Source: Prepared by the authors.

In the process of analyzing the textbooks, we established six categories that constitute the support that will help us reach our conclusions. The established categories are: **Variables, Intervals/Field of Variability, Correspondence/Field of Existence of the function, Examples, Elementary Functions/Graphical Representation and Exercises**. Each of them is present in the chapter where functions are studied, in each book analyzed, with a greater or lesser degree of depth, depending on the author's choice. The conclusions will be obtained by comparing the analysis of the listed categories with the ideas that permeated the reformist movement led by Felix Klein in order to identify from which perspective each author appropriated the theme of function, thus seeking an understanding of the affinity among them.

Felix Klein and functional thinking

It is possible to state that the beginning of the process of teaching the concept of function in elementary school arose from the concerns of the German mathematician Felix Klein. The idea that permeated this process is that of **Functional Thinking**. Christian Felix Klein, one of the most important mathematicians of the 19th century, was born on April 25, 1849 in Dusseldorf, then part of the Prussian Empire.

One of the most important references to Klein concerns his well-known Erlangen Program of 1872, when he considers geometry as the study of the properties of figures that remain invariant under a particular set of transformations. According to Eves,

[1] This collection was not mentioned by Oliveira Filho (2013) in his work.

> Appointed in 1872, at just 23 years of age, as a full professor at the Faculty of Philosophy and a member of the Council of the University of Erlanger, Felix Klein (1849-1925) prepared, according to custom, a lecture to introduce his new faculty colleagues and a written paper showing research interests in his mathematical field. The lecture, addressed to a large university auditorium, expressed Klein's pedagogical vision of the unity of all knowledge, an ideal that a complete education could not neglect for the sake of private studies. The written paper, which was distributed during the lecture, was intended for his peers in the department. Thus, the two parts of Klein's initial presentation revealed, on the one hand, his deep interest in pedagogical issues and, on the other, his serious involvement with mathematical research (EVES, 2004, p. 605).

His personal gifts as a talented mathematician, aware of the importance of the plural aspect of mathematics and, above all, his interest in teaching, ended up giving him a prominent role in the process of modernization in mathematics teaching. Despite this, his interest in issues related to teaching did not arise suddenly: they underwent a process of changing conceptions.

At first, he believed that the teaching of mathematics was justified by issues that were merely internal to mathematics itself, using a romantic view of this as justification (MIORIM, 1998). Still in relation to this phase, his enchantment with the virtues of mathematics itself led him to believe that its teaching should be present in the university education of all students of natural sciences and medicine (MIORIM, 1998). According to Miorim (1998, p.68), a turning point in Klein's vision was the period in which he worked at the Technisch Hochschule in Munich, between 1875 and 1880, when he participated in a group of mathematicians who were interested in the relation between science and technology. According to Miorim,

> In the early years of our century, Klein proposed a renewal of mathematics teaching based on changes in both secondary school and university studies. On the one hand, he advocated updating mathematics in secondary school, in order to be closer to the modern development of this area and also to the latest scientific and technological advances. On the other hand, he believed that the University should modify its teaching proposal, taking into account the needs of the future teacher (MIORIM, 1998, p.69).

This perspective was closely related to what Klein called “Double Discontinuity” and which we will discuss later.

From this change of direction in his thinking, two consequences can be inferred. One is related to the fact that, in order to change teaching in schools, an argument related to “psychological motives” was used, which he himself would explain:

> The teacher must be, so to speak, somewhat *diplomatic*; he must know the psychology of children in order to capture their interest, and this can only be achieved if he agrees to present things in an intuitive, easily assimilated way. Within the school, only in the upper classes can the doctrine be clothed in an abstract form [...], but this [...] should also extend to all education, even higher education; mathematics should always be presented in relation to everything that what could interest man and what he will use in his life. (KLEIN, *apud* MIORIM, 1998, p.69)

The other refers to the introduction of new content. The most important of these possibilities concerns the concept of function. From 1893 onwards, Klein began to draw the attention of the international community regarding reformulations in the teaching of mathematics, when he began to speak to teachers about the importance of Functional Thinking in school mathematics. In addition to speeches, the works published with Götting and Riecke (HAMLEY, 1934) stand out. According to Hamley (1934), in Breslau, Klein gave a lecture where he emphasized the importance of the concept of function. According to him,

> the concept of function presented graphically should form the central notion of teaching mathematics and that, as a natural consequence, the elements of calculus should be included in the curriculum of all nine-grade schools. (HAMLEY, 1934, p.52)

From this time onwards, the Breslauer Commission was established to formulate and define proposals for reform, especially with regard to the concept of function. According to Hamley (1934), the proposals of this commission were presented at a conference in Meran in 1905, in a form known as Meraner Lehrplan (in free translation, Meran Curriculum). According to Hamley (1934, p.53), these proposals aimed to:

> align the course of instruction more closely with the natural process of mental development than previously, develop, as far as possible, the faculty of contemplating natural phenomena from a mathematical point of view, and make the student increasingly aware of the continuity of the subject as he passes from one stage to another - a psychological, utilitarian and didactic principle. (HAMLEY, 1934, p.53)

In this same document, according to Hamley (1934), this intent would be achieved through a unifying principle that would allow this continuous change to occur. This principle was based on education from the perspective of Functional Thinking. The idea of unification was dear to Klein, ever since the Erlanger Program, when he attempted a unifying perspective of mathematics based on Group Theory.

From then on, Klein then focused his efforts on disseminating this teaching perspective, even going so far as to state, in the wake of these discussions, that the concept of function was not just a mathematical method, but the soul and heart of mathematical thought. He emphasized the possibilities of analyzing mathematics from a dynamic point of view, as opposed to teaching based on Euclid's synthetic geometry, which was very popular in Germany at the time (MIORIM, 1998).

Klein's ideas took on an international dimension from the moment the reformist wave reached the International Mathematical Congress, the first of which was held in 1897 in Zurich. After this congress, which in a way united the international mathematical community, issues related to the teaching of mathematics also began to be discussed. These issues, which had previously been dealt with in a localized manner, began to be discussed in a more global manner. However, according to Miorim (1998), the discussions did not bear fruit. This generated dissatisfaction and led to the creation of an international commission to study issues related to the teaching of mathematics. This commission was created after the IV International Mathematical Congress held in Rome in 1908, and was called the International Commission on the Teaching of Mathematics. Although he was not present at the congress, Felix Klein was appointed to chair the newly created commission. However, during the first meeting of the commission in September 1908, in the city of Cologne, the scope was expanded to all levels of mathematical instruction.

The work of the commission continued until 1914, when the First World War broke out. This event delayed the work of the committee until 1918, when the conflict ended. Despite this, according to Miorim (1998), activities related to the teaching of mathematics were not completely affected. The same perception felt by Klein years before that there was a phenomenon of "Double Discontinuity", referring to basic education and higher education, was a difficulty to be overcome, according to the understanding of the International Committee. In this sense, also according to Miorim (1998), the introduction of calculus in basic education was among the suggestions of the committee, in the post-war period. Within this context, the International Movement for the Modernization of Mathematics Teaching had Felix Klein as one of its greatest articulators.

The teaching of functions in Brazil

The insertion of the concept of function in Brazilian secondary education occurred through the creation of a new subject, in 1929, called mathematics, formed by the unification of three subjects, algebra, arithmetic and geometry, previously treated independently. This unification was motivated by the international movement to modernize secondary mathematics education, led by Felix Klein.

In 1912, the 5th International Mathematics Congress took place and Eugênio Barros Raja Gabaglia was appointed as Brazil's representative. However, his trip did not bring about any concrete changes. At the same time, Professor Euclides Roxo was invited to teach at *Colégio Pedro II* in 1915. Roxo was the great mentor of the modernization movement in Brazil, and he encountered much resistance to his innovative ideas, which ended up creating difficulties for discussions to take place within *Colégio Pedro II* about what was being proposed. Among the opponents were Professors Raja Gabaglia and Joaquim Inácio de Almeida Lisboa. With Gabaglia's death in 1919 and his temporary leave from Lisbon during the last three years of the 1920s, Roxo's situation began to become a little more comfortable.

In 1923, Roxo published his book, "*Lições da Aritmética*", which was considered an important step towards his goals. In this book, Roxo already expressed his ideas, albeit in small outlines. In 1925, Roxo took over as director of the *Externato do Colégio Pedro II*. His proposal for the unification of the mathematics curriculum was approved in the *Ata da Congregação do Colégio Pedro II* on November 14, 1927, after much discussion. This unification, [...] was necessary to meet mainly two modernizing concepts. The first of these referred to the demands of establishing connections among the various branches of school mathematics. (...) The second concept delegated to the notion of function with its algebraic, geometric and tabular representations the role of coordinating the various subjects of secondary mathematics. (BRAGA, 2006, p.69)

On January 12, 1927, Roxo On the 29th, Decree 18564 was signed, formalizing the modernizing proposals defended by Roxo. Also in 1929, Euclides Roxo published the book "*Curso de Matemática Elementar*", volume 1, written with the ideas of the new program and a new teaching method.

In the wake of the Revolution of 1930, in 1931, a broad reform of Brazilian education was decreed, the Francisco Campos Reform. As the first minister of the newly created Ministry of Education and Health, Francisco Campos divided secondary education into two cycles: five-year elementary education and two-year education aimed at preparing for higher education. When Vargas assumed the presidency, Roxo resigned from his position as director of the *Externato do Colégio Pedro II*, but in the same year he was appointed by Vargas and Campos as director of the *Internato do Colégio Pedro II* and was invited to participate in the organization of a reform for Brazilian education. However, all of his proposals for innovation in secondary education soon came under fire. Upon his return from Lisboa, an intense debate ensued between

the two. Amidst the discussions, Roxo highlighted the theme of functions in one of his articles, "the concept of function as the axial axis of teaching". According to Rocha (2001, apud Valente, 2002, p.19), Roxo's intention was to "familiarize students from an early age with the notion of function, through its graphic and analytical representation, and to make it the central point of teaching, in order to enable the connection among the different parts of mathematics."

Thus, the concept of function was included in school mathematics in the programs resulting from the Francisco Campos Reform. This presence, from the first cycle, was ensured by a rigid and authoritarian educational policy.

In 1934, the Ministry of Education and Health was taken over by Gustavo Capanema. In January 1936, Capanema decided to conduct a survey about education in Brazil, distributing an extensive and detailed questionnaire. In addition to this, he carried out many other research projects in an attempt to reorganize the national education system. The Capanema Reform was enacted on April 9, 1942, through Decree 4,244. This reform aimed to reorganize the division of secondary education, continuing with two cycles. However, the first would have four cycles, known as the *ginasial* course, and the second cycle, with three years, would be the *colegial* course, in two modalities: classical and scientific. The Ministerial Order created a committee to organize the teaching program, with Capanema as president and Roxo as one of the members. There were also representatives of the Church, Arlindo Vieira, and of the military, Ignácio Azevedo Amaral.

On May 20, 1942, Roxo sent a letter to Capanema with his proposal for the teaching programs in the *ginasial* course. Upon receiving this proposal, Capanema sent it to Vieira and Amaral. Arlindo Vieira makes several suggestions, and the only exclusion he defends is Roxo's main idea. For him, including the notion of function in the *ginasial* course would be confusing for the students at that time. On the other hand, the military demonstrated agreement with the proposals suggested by Roxo. On June 11, 1942, the ministerial decision accepted Arlindo Vieira's suggestion and the notion of function was removed from *ginasial* education, that is, from the first cycle of secondary education.

In the 1950s, the Simões Filho Reform came into effect and was conducted by executive orders 966, of October 2, 1951, and 1045, of December 14, 1951. This reform introduced the so-called "*Programas Mínimos*" to secondary education in Brazil. This program was intended to establish a minimum limit of content, which all educational institutions would be required to implement. In addition to advocating the reorganization of the mathematics curriculum in secondary education, the Simões Filho Reform proposed a teaching program with 3 (three) hours per week for the subject of Mathematics. According to ordinance 1045, there were some differences in the content between the classical and scientific courses, however the concept of function would be seen in both courses and only in the 3rd grade. In view of the establishment of this new program for mathematics, the director of the *Externato do Colégio Pedro II*, professor Gildásio Amado, comments that

> Secondary education programs were prepared by committees appointed by the Minister of Education. In 1951, the Congregation of *Colégio Pedro II* demanded that it itself create the programs for the subjects taught at the school. Minister Simões Filho went further: he determined that the Programs of *Colégio Pedro II* would be official for all secondary schools in the country. The programs sent to the Minister – said the president of the Congregation – "contain the minimum material and allow secondary school teachers to find a disciplinary guide, without prejudice to the freedom to present the subjects in accordance with didactic conveniences". They were, therefore, "*programas mínimos*". (AMADO, 1973 apud VALENTE, 2008, p.20)

However, the "*Programas Mínimos*" were not successful and were heavily criticized by several textbook authors. Thus, we can see that there were major movements in the teaching of mathematics during this period, many of them involving the inclusion of the concept of function. A little later, with the emergence of the MMM, the ways in which mathematical content was applied began to be questioned and the perspective adopted for teaching functions incorporated the ideas linked to this movement, adopting a view predominantly based on Set Theory.

Description of the collections

We will present a description of the seven collections of textbooks analyzed, referring to the third grade of *colegial.* In addition to the structural description of the book, we highlight external characteristics and present a brief biography of the authors.

Collection 1: Euclides Roxo; Roberto Peixoto; Haroldo Cunha and Dacorso Netto.

The Mathematics 2nd Cycle collection, also known as the "*Livro dos Quatro Autores*", had Euclides Roxo as one of its main creators. The person responsible for the content of Algebra was Haroldo Lisboa da Cunha, who, in 1934, took over the Mathematics Chair at *Colégio Pedro II*. He was a professor at the State University of Guanabara (UEG, now the State University of Rio de Janeiro, UERJ) and in 1960 became rector (Silva and Bernardino, 2019). Professor Cesar Dacorso Netto was a professor at the Institute of Education, having authored some publications aimed at higher education. Finally, we have Professor Roberto Peixoto, who also taught at the Institute of Education. According to Bernardino,

> This statement indicates that the changes to the edition were made by only two of the four authors of the work. Possibly, when adapting the work to the program established by ordinance 90 of 1951, the authors were unable to maintain the initial division of the preparation of the edition of the work among themselves. Among the reasons, we can highlight the death of Euclides Roxo in 1950. Cesar Dacorso Netto also did not participate in the preparation of the 1955 and 1956 editions; however, we did not find any information that could have caused his withdrawal from the reorganization of the series. (BERNARDINO, 2016, p. 89-90).

The book has no preface or introduction. Its programmatic content is distributed according to ordinance 1045, of December 14, 1951, which determines the *Programas Mínimos*. The fact that the contents are not dissociated raises the hypothesis that the work is the combination of three books: algebra, geometry and analytical geometry. According to Bernardino,

> This conjecture that the book is made up of three distinct works makes sense when we look at the time in which the work was written. The Mathematics 2nd Cycle series was published shortly after the Capanema reform came into effect. (..) Knowing in advance how the reform would affect education, he set about creating a collection in order to stand out in the publishing market. In the urgency of creating and publishing the collection, Euclides Roxo established a partnership with three other authors who, coincidentally, studied different areas of mathematics. (BERNARDINO, 2016, p. 81).

This is certainly one of the most notable collections, since it is the oldest and also has as one of its authors Professor Euclides Roxo who, as already mentioned, played an important role in the reformulation of mathematics teaching in Brazil.

Collection 2: Edison Farah, Omar Catunda, João Batista Castanho and Benedito Castrucci.

The book "*Matemática, 3ª série, curso collegial*" was also written by four authors: Edison Farah, Omar Catunda, João Batista Castanho and Benedito Castrucci. The authors have in common the fact that they were members of the faculty of the *Faculdade de Filosofia Ciências e Letras* (FFCL) at USP. Edison Farah was a doctoral student of Professor Omar Catunda, and was appointed Interim Professor of Higher Analysis also at FFCL-USP. In 1942, Professor Benedito Castrucci assumed the Chair of Analytical, Projective and Descriptive Geometry at the same institution, and in 1943 defended his doctoral thesis, with Omar Catunda as one of the members of the committee (Moraes, p. 59-60, 2008). Professor Omar Catunda was hired by FFCL-USP as an assistant to Luigi Fantappiè in the Mathematical Analysis discipline. After studying for a period at the University of Rome, he returned to Brazil and was appointed interim professor responsible for the chair of Mathematical Analysis and Higher Education and appointed Head of the Department of Mathematics at FFCL-USP, where he became a full professor after defending his thesis. He concluded his professional career in Bahia, where he took on the role of Director of the *Instituto de Matemática e Física da Universidade Federal da Bahia* (IMFUFBa) (Duarte, 2008). Professor João Batista Castanho was also an assistant professor in the discipline Complements of Mathematics for the courses of Chemistry, Social Sciences and Pedagogy, also at FFCL-USP. In 1950 he defended his doctoral thesis under the supervision of Professor Fernando Furquim.

This collection seemed little known to us, since it was not even mentioned in Oliveira Filho (2013). Observing its structure, we notice that the book presents the programs for the third year of *colegial* before presenting the general index. The concept of functions, presented in unit II of volume 3, was the responsibility of professor Omar Catunda. Also in the preface, the authors mention that "*We respected the guidelines of the official program (3rd year), adding only the complements indispensable for the good development of the arguments discussed".*

Collection 3: Thales de Faria Mello Carvalho

The book "Matemática para os cursos Clássico e científica, terceira ano" (Matematics for the Classical and Scientific Courses, Third Year) was written by Thales de Faria Mello Carvalho. Born in the city of Rio de Janeiro on April 22, 1915, he had extensive experience in teaching, teaching at the secondary school of the Institute of Education and at the *Faculdade de Ciências Ecnômicas da Universidade do Brasil*. When it comes to the development of mathematics textbooks, this work is one of the most important. According to Brum and Silva,

> This publication came about after the secondary education reform promoted by Minister Gustavo Capanema in 1942, seeking to comply with the ministerial order of March 1943, which set out the official programs for the 2nd cycle of secondary education. (BRUM and SILVA, 2022, p. 225).

In 1969, the three volumes were brought together in a single book and published by the *Fundação Getúlio Vargas*. (Brum and Silva, 2022). We can highlight that the volume intended for the third year dedicates its first chapter to the concept of sets and the idea of correspondence. Here, we can already see the introduction of terms related to set theory.

Collection 4: Jairo Bezerra

Professor Manoel Jairo Bezerra began his professional career at Colégio Metropolitano, located in the city of Rio de Janeiro, and was a professor at the *Escola de Comando e estado Maior da Aeronáutica*, *Colégio Pedro II*, *Colégio Naval* and *Curso de Técnica de Ensino do Exército*. He was also a Professor at the *Instituto de Educação* and a professor and owner of a *Curso Pré-Normal* that bore his name (MACIEL, 2012).

According to Bezerra (2010, apud MACIEL 2012), among his collections of textbooks, the most notable is *Curso de Matemática* for the first, second and third years of the classical and scientific courses. Initially published in three volumes, it was probably the first Brazilian didactic work, at least with regard to the teaching of Mathematics, in a "single volume" format. According to Bigode and Valente (2003), when asked about the idea of merging the three volumes into a single volume, he stated that "The main idea, according to Jairo, was to streamline the theory and present many solved exercises" (BIGODE and VALENTE, 2003, p.6). Jairo Bezerra considered that the sales success of his book was not due to the didactics itself, but to the way the exercises were presented, with immediate applications after the theoretical presentation. When asked about his references for writing the book, Jairo stated that he was "afraid of straying from what Professor Thales de Mello said". He considered using the concepts of that author and his ideas for presenting the lessons. (BIGODE and VALENTE, 2003, p. 9-10). In our analyses, we considered the work as a single volume.

Collection 5: Ary Quintella.

Ary Norton de Murat Quintella was a teacher at the *Colégio Militar do Rio de Janeiro* since 1937. With a long career in education, Quintella was also a teacher at the *Instituto de Educação* from 1950 to 1960. His text is considered one of the educational bestsellers of its time and he was one of the great names of *Companhia Editora Nacional* (VALENTE, 2008). His Mathematics collection is composed of three volumes, one for each series of classical and scientific courses.

On the back cover of the volume for the third year of *colegial*, the author states that the work is in accordance with the new *Programas Mínimos*, according to Ordinance 966, of 10/20/1951 and 1045 of 12/14/1951. The concept of function is presented in the first chapter of the book and is covered from intervals to graphical representation of functions. After the general index and exercise index, the book presents, as well as other collections analyzed, the Mathematics Programs for the third year of *colegial*. (VALENTE, 2008)

Collection 6: Algacyr Munhoz Maeder

Algacyr Munhoz Maeder was born on April 22, 1903, in the city of Curitiba, and began his studies there. Later, he moved to São Paulo and enrolled at *Colégio São Bento*. He returned to Curitiba to complete high school and finally enroll in the Faculty of Engineering at the *Universidade Federal do Paraná*, where he obtained the title of Civil Engineer. (LONGEN, 2007). According to Longen,

> Algacyr Munhoz Maeder was the author of 28 Mathematics books for Brazilian school education. Starting in 1928, still in the era of compendiums, he published four collections focused on teaching this subject that were published until 1962. These books testify to the transition between a compendium and a textbook, in addition to being records of the birth of the Mathematics discipline as a unification of its branches. They were written amidst reforms, decrees and teaching ordinances that occurred in Brazil. (LONGEN, 2007, p. xi).

The book "*Curso de Matemática, 3º ano colegial*" had a total of 8 editions. From the 1st to the 3rd edition, the publications followed Ministerial Ordinance no. 177, of March 16, 1943. The second edition was released in 1949 and, in its index, references are made to the contents according to the course for which it is intended, classical or scientific.

Collection 7: Alberto Nunes Serrão.

Alberto Serrão was a full professor of the Chair of Infinitesimal Calculus, Analytical Geometry and Notions of Nomography at the *Escola Nacional de Engenharia*. He is a civil engineer and geographer from the *Escola Nacional de Engenharia*. Former head professor of the Mathematics section at the *Colégio Universitário da Universidade do Brasil*. Former professor of Mathematics at the Complementary Course at *Colégio Pedro II*, of the *Instituto de Educação do Rio de Janeiro*. The first edition of the book "Algebraic Analysis" was released in 1940. The second edition, released five years later, in addition to a brief biography of the author, includes the target audience for which the volume was designed: "students of science courses, military schools, candidates for entrance exams for engineering, chemistry and architecture schools and students applying to philosophy schools". In addition to this information, the number of solved exercises is also listed, 723.

In the preface to the book, some information is quite pertinent. The author lists some differences between the first and second editions of the volume, among them the increase of about 250 pages, caused by the way some content was treated and by the inclusion of numerous exercises.

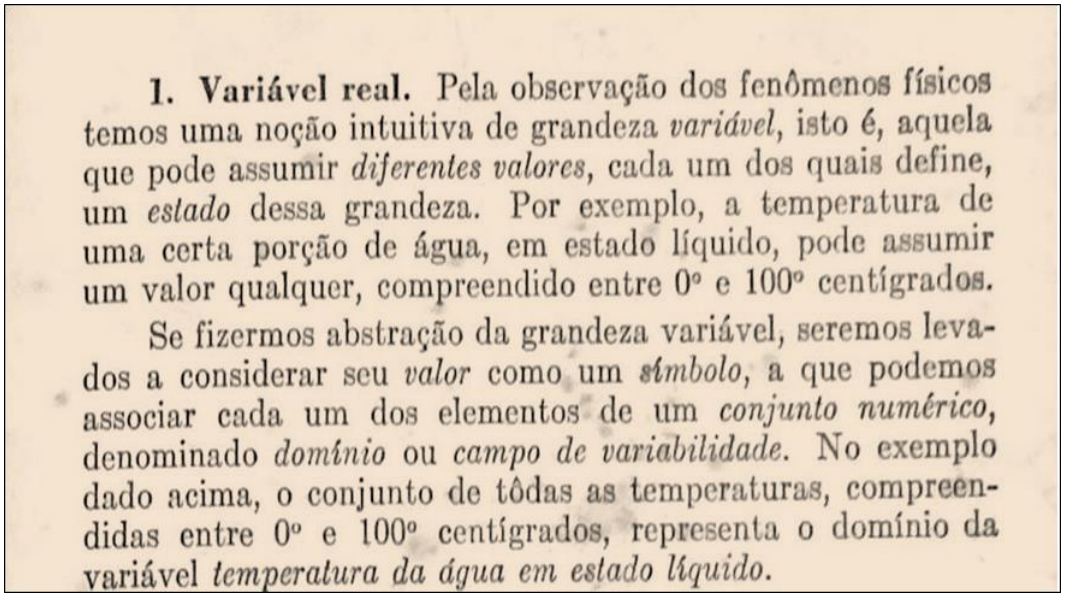
The chapter on functions of one variable. Limits. Continuity from the first edition is now divided into three others, treated in an extensive manner. Finally, the author states that "all the modifications made were always aimed at improving the compendium in general, making it clearer and more precise without in any way losing its character as an eminently didactic work".

Analysis of categories

The categories listed represent, in general terms, the sequence in which the theme of functions is presented in the books of the collections analyzed. Since they are closely linked to the process that leads to an introduction to calculus, they alone serve to indicate to us that the sense of appropriation of Klein's ideas, as they relate to Functional Thinking, were directed towards this end.

The idea of variable is associated with a symbol that represents elements of a given set. In the case of numerical sets, it is a letter that designates the numerical value of a certain quantity. It is worth noting here that collection 4 does not mention the definition of variables and that collection 5 mentions the terms continuous and progressive variable. Collection 3 indicates a physical motivation in the definition of a variable, as shown in the excerpt presented in Figure 1. Here is an important element to highlight in this collection: from the moment a physical motivation is used, we understand, at least implicitly, that the author had already envisioned an idea of variation, growth, themes dear to Differential and Integral Calculus and, within a perspective pointed out by the reformist movement (GÜTZMAN, 1708, apud KRÜGER, 2019, p. 37).

Figure 1 – Example of a constant variable in collection 3.



1. Variável real. Pela observação dos fenômenos físicos temos uma noção intuitiva de grandeza *variável*, isto é, aquela que pode assumir *diferentes valores*, cada um dos quais define, um *estado* dessa grandeza. Por exemplo, a temperatura de uma certa porção de água, em estado líquido, pode assumir um valor qualquer, compreendido entre 0° e 100° centígrados.

Se fizermos abstração da grandeza variável, seremos levados a considerar seu *valor* como um *símbolo*, a que podemos associar cada um dos elementos de um *conjunto numérico*, denominado *domínio* ou *campo de variabilidade*. No exemplo dado acima, o conjunto de tôdas as temperaturas, compreendidas entre 0° e 100° centígrados, representa o domínio da variável *temperatura da água em estado líquido*.

Source: Excerpt from Volume 3 of Collection 3, page 31.

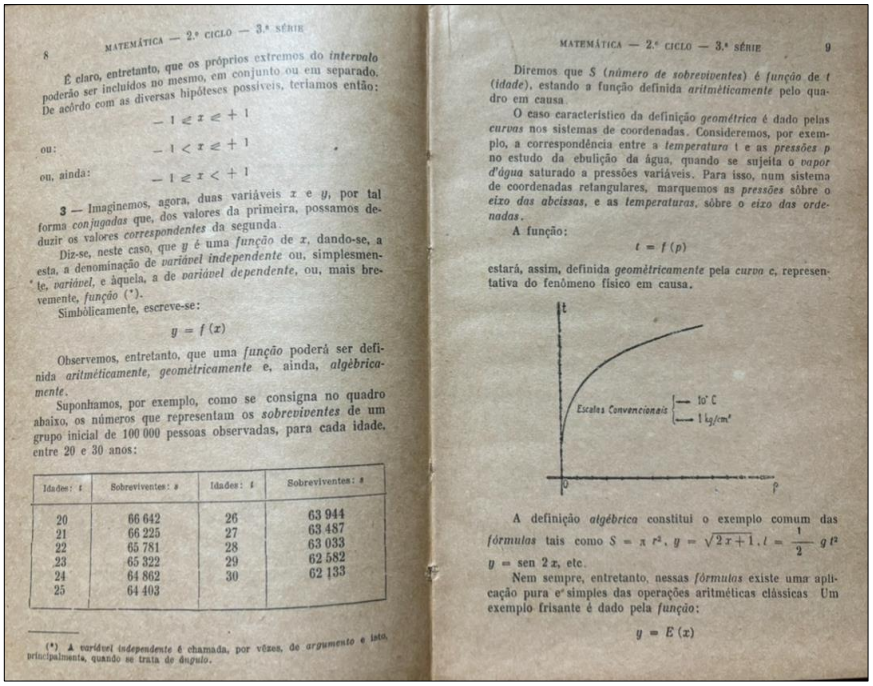
In general, the definition of interval is not considered in the collections analyzed, at least not so emphatically. Some collections do not even mention it, while others do so after the definition of function. Collection 3, for example, defines intervals in a previous chapter when working with numerical sets. It is important to highlight that collection 5 uses the notion of epsilons[2] to define intervals. The concept of variability field is explored only in collections 6 and 7.

From all the categories listed, perhaps the most important is **correspondence**, since it is the central object in the definition of function. Collections 1, 2 and 6 do not mention the uniqueness of the correspondence law, but use the term "well-determined" or "determined number", leaving this information implicit. Collection 3 is the only one that leaves no room for questioning. Collections 6 and 7 suggest that every function is surjective, since they define the concept of function from the domain over its image. We highlight in the definition given by collection 4 that the author qualifies function as a law that makes a variable correspond to one or more variables. From this we conclude that there is a variety of meanings in which the authors seem to give the idea of correspondence, which corroborates the thinking of Chartier (1990, p.26-27). Closely linked to the idea of correspondence is the idea of the **Function's field of existence.** It is rarely mentioned in the collections. Collection 3 defines it only after the examples. Collection 6 defines domain together with the definition of function and collection 7 considers it as a real numerical set that the variable can assume. Once again, a plurality of meanings is indicated, Biccas (2012) and Choppin (2002, p.15).

We highlight that, after the definition of function, collections 2, 3, 4 and 5 present examples involving essentially analytical expressions, which is in accordance with Roxo's ideas (ROCHA, 2001, apud VALENTE, 2002, p.19), with emphasis on collection 3, which presents an example where the idea of correspondence is emphasized. Collections 6 and 7 do not present examples. Collection 1 presents three contextualized examples, using algebraic, arithmetic and geometric representation, as shown in figure 2.

[2] The epsilon notation we refer to concerns the treatment of intervals based on the idea of neighborhood of a point, a topic that belongs to the most advanced studies of Differential and Integral Calculus. This element, in itself, also serves as an indication of a future intention in the study of functions.

Figure 2 – Various forms of representing a function



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É claro, entretanto, que os próprios extremos do *intervalo* poderão ser incluídos no mesmo, em conjunto ou em separado. De acôrdo com as diversas hipóteses possíveis, teríamos então:

$$-1 \leqslant x \leqslant +1$$

ou: $$-1 < x \leqslant +1$$

ou, ainda: $$-1 \leqslant x < +1$$

3 — Imaginemos, agora, duas variáveis x e y, por tal forma *conjugadas* que, dos valores da primeira, possamos deduzir os valores *correspondentes* da segunda.

Diz-se, neste caso, que y é uma *função* de x, dando-se, a esta, a denominação de *variável independente* ou, simplesmente, *variável*, e àquela, a de *variável dependente*, ou, mais brevemente, *função* (*).

Simbòlicamente, escreve-se:

$$y = f(x)$$

Observemos, entretanto, que uma *função* poderá ser definida *aritmèticamente*, *geomètricamente* e, ainda, *algèbricamente*.

Suponhamos, por exemplo, como se consigna no quadro abaixo, os números que representam os *sobreviventes* de um grupo inicial de 100 000 pessoas observadas, para cada idade, entre 20 e 30 anos:

Idades: t	Sobreviventes: s	Idades: t	Sobreviventes: s
20	66 642	26	63 944
21	66 225	27	63 487
22	65 781	28	63 033
23	65 322	29	62 582
24	64 862	30	62 133
25	64 403		

(*) A *variável independente* é chamada, por vêzes, de *argumento* e isto, principalmente, quando se trata de *ângulo*.

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Diremos que S (*número de sobreviventes*) é *função* de t (*idade*), estando a função definida *aritmèticamente* pelo quadro em causa.

O caso característico da definição *geomètrica* é dado pelas *curvas* nos sistemas de coordenadas. Consideremos, por exemplo, a correspondência entre a *temperatura* t e as *pressões* p no estudo da ebulição da água, quando se sujeita o *vapor d'água* saturado a pressões variáveis. Para isso, num sistema de coordenadas retangulares, marquemos as *pressões* sôbre o *eixo das abcissas*, e as *temperaturas*, sôbre o *eixo das ordenadas*.

A função:

$$t = f(p)$$

estará, assim, definida *geomètricamente* pela *curva c*, representativa do fenômeno físico em causa.

A definição *algébrica* constitui o exemplo comum das *fórmulas* tais como $S = \pi r^2$, $y = \sqrt{2x+1}$, $l = \frac{1}{2} g t^2$ $y = \text{sen } 2x$, etc.

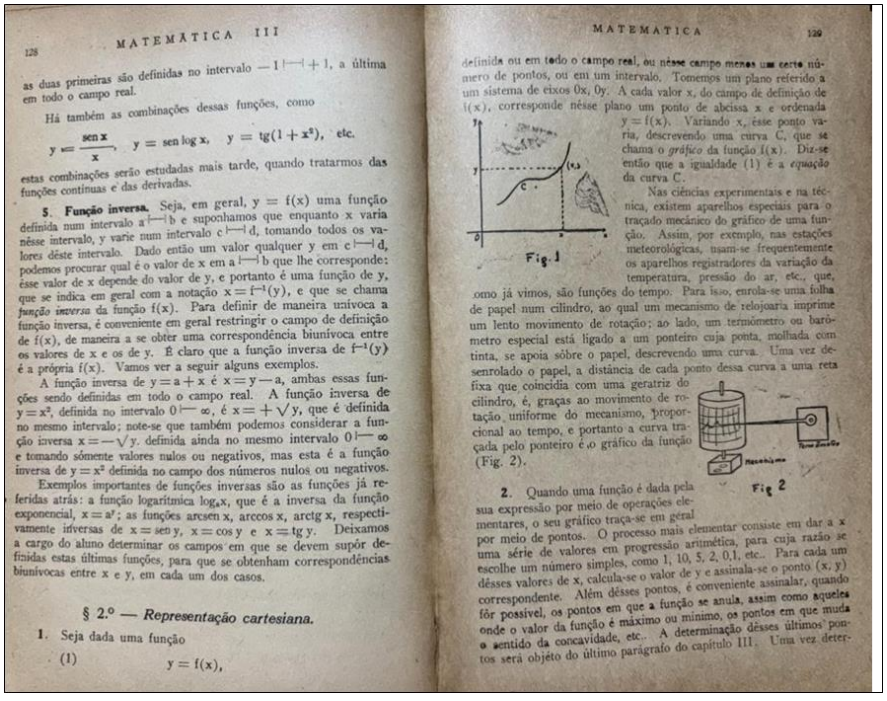
Nem sempre, entretanto, nessas *fórmulas* existe uma aplicação pura e simples das operações aritméticas clássicas. Um exemplo frisante é dado pela *função*:

$$y = E(x)$$

Source: Excerpt from volume 3, Collection 1, pages 8 and 9.

The **graphical representation** of functions is done in a very intuitive way, since the idea that guides the authors is that the graph of a function can be drawn by marking some points on the plane. Some examples that can be described through this idea are explored in collections 1 and 3. An interesting highlight occurs in collection 2, where the authors use a model of a temperature recording device in order to make the graphical representation of a function more concrete, where we identify a perspective of contemplating a natural phenomenon, in the sense indicated by Krüger (2019, p.37).

Figure 3 – Resource used to visualize the idea of a graph of a function.



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as duas primeiras são definidas no intervalo $-1 \vdash\dashv +1$, a última em todo o campo real.

Há também as combinações dessas funções, como

$$y = \frac{\text{sen } x}{x}, \quad y = \text{sen} \log x, \quad y = \text{tg}(1+x^2), \text{ etc.}$$

estas combinações serão estudadas mais tarde, quando tratarmos das funções contínuas e das derivadas.

5. Função inversa. Seja, em geral, $y = f(x)$ uma função definida num intervalo $a \vdash\dashv b$ e suponhamos que enquanto x varia nêsse intervalo, y varie num intervalo $c \vdash\dashv d$, tomando todos os valores dêste intervalo. Dado então um valor qualquer y em $c \vdash\dashv d$, podemos procurar qual é o valor de x em $a \vdash\dashv b$ que lhe corresponde: êsse valor de x depende do valor de y, e portanto é uma função de y, que se indica em geral com a notação $x = f^{-1}(y)$, e que se chama *função inversa* da função $f(x)$. Para definir de maneira unívoca a função inversa, é conveniente em geral restringir o campo de definição de $f(x)$, de maneira a se obter uma correspondência biunívoca entre os valores de x e os de y. É claro que a função inversa de $f^{-1}(y)$ é a própria $f(x)$. Vamos ver a seguir alguns exemplos.

A função inversa de $y = a + x$ é $x = y - a$, ambas essas funções sendo definidas em todo o campo real. A função inversa de $y = x^2$, definida no intervalo $0 \vdash\dashv \infty$, é $x = +\sqrt{y}$, que é definida no mesmo intervalo; note-se que também podemos considerar a função inversa $x = -\sqrt{y}$, definida ainda no mesmo intervalo $0 \vdash\dashv \infty$ e tomando sòmente valores nulos ou negativos, mas esta é a função inversa de $y = x^2$ definida no campo dos números nulos ou negativos.

Exemplos importantes de funções inversas são as funções já referidas atrás: a função logarítmica $\log_a x$, que é a inversa da função exponencial, $x = a^y$; as funções arcsen x, arccos x, arctg x, respectivamente inversas de $x = \text{sen } y$, $x = \cos y$ e $x = \text{tg } y$. Deixamos a cargo do aluno determinar os campos em que se devem supôr definidas estas últimas funções, para que se obtenham correspondências biunívocas entre x e y, em cada um dos casos.

§ 2.º — *Representação cartesiana.*

1. Seja dada uma função

(1) $$y = f(x),$$

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definida ou em todo o campo real, ou nêsse campo menos um certo número de pontos, ou em um intervalo. Tomemos um plano referido a um sistema de eixos 0x, 0y. A cada valor x, do campo de definição de f(x), corresponde nêsse plano um ponto de abcissa x e ordenada $y = f(x)$. Variando x, êsse ponto varia, descrevendo uma curva C, que se chama o *gráfico* da função f(x). Diz-se então que a igualdade (1) é a *equação* da curva C.

Fig. 1

Nas ciências experimentais e na técnica, existem aparelhos especiais para o traçado mecânico do gráfico de uma função. Assim, por exemplo, nas estações meteorológicas, usam-se frequentemente os aparelhos registradores da variação da temperatura, pressão do ar, etc., que, como já vimos, são funções do tempo. Para isso, enrola-se uma folha de papel num cilindro, ao qual um mecanismo de relojoaria imprime um lento movimento de rotação; ao lado, um termômetro ou barômetro especial está ligado a um ponteiro cuja ponta, molhada com tinta, se apoia sôbre o papel, descrevendo uma curva. Uma vez desenrolado o papel, a distância de cada ponto dessa curva a uma reta fixa que coincidia com uma geratriz do cilindro, é, graças ao movimento de rotação uniforme do mecanismo, proporcional ao tempo, e portanto a curva traçada pelo ponteiro é o gráfico da função (Fig. 2).

Fig. 2

2. Quando uma função é dada pela sua expressão por meio de operações elementares, o seu gráfico traça-se em geral por meio de pontos. O processo mais elementar consiste em dar a x uma série de valores em progressão aritmética, para cuja razão se escolhe um número simples, como 1, 10, 5, 2, 0,1, etc.. Para cada um dêsses valores de x, calcula-se o valor de y e assinala-se o ponto (x, y) correspondente. Além dêsses pontos, é conveniente assinalar, quando fôr possível, os pontos em que a função se anula, assim como aqueles onde o valor da função é máximo ou mínimo, os pontos em que muda o sentido da concavidade, etc.. A determinação dêsses últimos pontos será objeto do último parágrafo do capítulo III. Uma vez deter-

Source: Extracted from Volume 3, Collection 2, pages 128 and 129.

The **elementary functions** are polynomial, trigonometric, transcendental functions and some more specific examples that result from the combination of these. The objects that give rise to these functions, namely polynomials, sine, cosine, tangent, as well as powers and exponents are dealt with in previous volumes of the collections, as in the case of collections 1, 2, 3, 5 and 6. In book 4, due to its characteristic of being a single volume, references are made within the same book, but to previous chapters. In the case of book 7, it is assumed that the reader already has this notion from other experiences. This is justifiable, since the book is aimed at an audience that, presumably, has already been to school. Here we saw little affinity between the authors' form of appropriation and the ideals of the reformist movement.

The **exercises** vary greatly from collection to collection, both in number and in emphasis on what the author considers most important for the reader to master. However, most of them place a high degree of importance on determining the Definition Field of a function, manipulating analytical expressions, and sketching the graph of some of them, which, in the latter case, is surprising, since, as previously pointed out, rigorous treatment can only be done through the study of differential calculus.

Some insights from the analysis of categories

The Functional Thinking seems to have been a point of great importance and interest for Felix Klein with regard to teaching. Its unifying character as a mathematical object may have strongly influenced this belief (KRÜGER, 2019, p.38). In addition, the idea of Functional Thinking encompassed other principles: the psychological and the utilitarian (KRÜGER, 2019, p.38). The synthesis of this thought is embodied in Schotten's speech, brought by Krüger:

> The Meraner Lehrplan chose the concept of function as a binding element. This concept had become popular in algebra or arithmetic and was not foreign to schools.... But the systematic application of all school mathematics based on this idea was lacking. (LIETZMANN, 1926, p. 231).

Based on the data obtained from the analysis of the collections and comparing them with the perception of Functional Thinking in force at the time, mainly from the perspective of Krüger (2019), we were led to conclude that the way in which Klein's proposals were appropriated by Brazilian textbook authors in the period in question, when viewed with the intention of including them at the high school level, promotes an attempt to place them as prolegomena to the study of Calculus. There is a coincidence between this perception and that which permeated the thinking at the time of the reformist movement (Krüger, 2019, p. 44). Despite this, the prospect of including calculus would occur, for the reformists, at a less intense pace than in a university course. According to Krüger,

> The new subject on differential and integral calculus should not act as an additional subject on the curriculum. Instead, an "organic" structure of school mathematics would be achieved by emphasizing calculus as a culminating point of higher mathematics education. Thinking about functional variations and dependencies should be practiced and made flexible in order to prepare for learning calculus. Education in the habit of Functional Thinking can, therefore, be considered as an attempt to establish a propaedeutics of calculus in high schools. (KRÜGER, 2019, p. 45).

In the case of Brazilian authors, a fact that reinforces the belief in the position of the study of functions concerns the very location of this content in the books. In all the collections analyzed, this theme is found in the same volume that includes the principles of Differential and Integral Calculus and Analytical Geometry, which, according to Tobies (2019, p.15), was in agreement with Klein's ideas and which here takes on an official character based on the *Programas Mínimos*. In some cases, the themes are interspersed with each other, as in the case of collection 1. The very way of approaching the content of functions presented in the books – based on the categories we listed and which have little affinity with the original proposals – seems to indicate a similarity to the presentation of functions in specific books on Differential and Integral Calculus, intended for higher education. In our understanding, this may be related to the fact that, in at least 5 (five) collections, the authors were linked to higher education.

Another fact concerns the somewhat summarized treatment that the authors give when dealing with specific examples such as elementary functions. In fact, the idea of functional dependence, which would reveal an opportunity to relate them to natural phenomena (GUTZMER, 1908, *apud* KRÜGER, 2019, p.37), appears little, except in some isolated examples, with the authors' attention basically focused on the analytical expression.

This ends up giving functions, such as logarithmic, exponential and trigonometric functions, an essentially algebraic treatment, since some authors often refer the discussion to previous volumes of the collection where the objects associated with them were presented as objects *per se*, meaning that, when looking at them from the perspective of Functional Thinking, the treatment is done in a very superficial manner, often in a review tone, as in the case of collection 7. Furthermore, the same fact occurs with several of its elements, such as graphical representation. This fact corroborates what we pointed out previously: the authors seem to have been greatly influenced by the presentation of functions made in some works about Calculus.

There are also peculiarities in the appropriation by national authors. One of them concerns the inclusion of themes such as the duality between multivocal or plurivocal and univocal functions, in which the authors' appropriation by those who served as references was not at all complete to the point of giving them a real meaning of this dualism and the real need to present it in a textbook for high school. In fact, this theme, whose presentation appears without any justification and which is not even mentioned later, ends up being isolated within each collection.

Another peculiarity indicates that the way in which Brazilian authors viewed the concept of function was also linked to what Klein called “Double Discontinuity”. There was a discomfort on his part with the teaching of mathematics, materialized in what he called “Double Discontinuity”. One of them was that the mathematics learned in secondary school was not adequate to meet the demands of what was taught in universities or technical schools and, according to Klein himself, translated by Miorim (1998, p. 60):

> The young student found himself, upon beginning his [university] studies, faced with problems that reminded him nothing of the things that had occupied him until then, and therefore he immediately and completely forgot all of them" (KLEIN, 1927, v. 1, p. 1.).

The other refers to teacher training, which lacked a greater connection with the knowledge that he would need to fulfill his mission in the classroom. Klein himself, in a translation of Miorim (1998, p.60), states that the newly graduated teacher would retain few or no fruitful memories of his training, since from traditional teaching and university studies “only a more or less pleasant memory would remain, but that would not exert even the slightest influence on his performance in teaching”. According to Hamley, Klein states that

> The elements of infinitesimal calculus, properly treated, provide much more suitable material for mathematical education in schools than that heterogeneous and lifeless subject, which today is so repellent to boys who have no special ability for mathematics. Furthermore, calculus is indispensable for a clear understanding of numerical physical phenomena and, from the point of view of mental training, it is an essential element of mathematical education. (HAMLEY, 1934, p.54)

According to our analyses, Brazilian authors attempted to solve, at least in one direction, this problem identified by Klein.

Conclusions

From the study above, we were able to verify that there were two types of manifestation of the idea of Functional Thinking in Brazilian secondary education, based on some textbooks that circulated in the period prior to the MMM. One was more linked to the search for the introduction of this theme, based on the idea of "Functional Thinking", as can be seen in the research of Braga (2003) and Oliveira (2009). The other aimed to use it as a backdrop to present a proposal to partially solve what Klein called the phenomenon of "Double Discontinuity". In the future, we intend to deepen the analyses regarding the presentation of functions in other periods that encompass both the MMM and the period in which the movement presented a deflection.

References

BERNARDINO, Camila, Libanori. *Números complexos: Um estudo histórico sobre sua abordagem na coleção Matemática 2º ciclo.* 2016. Dissertação. (Mestrado em Educação Matemática) - Universidade Estadual Paulista, 2016.

BICCAS, Maurilane de Souza. Roger Chartier: contribuições para a história da educação. In: LOPES, E. M. T.; FARIA FILHO, L. M. de.; (Org.). *Pensadores sociais e história da educação II.* 2ed.Belo Horizonte: Autêntica Editora, 2012, v. 1, p. 269-296.

BIGODE, Antonio José Lopes & VALENTE, Wagner Rodrigues. (2003). O Tijolão, o Bezerrão: histórias de Jairo Bezerra, histórias da Educação Matemática**.** *Revista da Sociedade Brasileira de Educação Matemática,* [s.l.]. Ano X, n. 13, 4-12.

BITTENCOURT, Circe Maria Fernandes. *Livro didático e saber escolar (1810-1910).* Belo Horizonte: Autêntica, 2008, p. 239.

BRAGA, Ciro. *O processo inicial de disciplinarização de função na matemática do ensino secundário brasileiro.* 2003. Dissertação (Mestrado em Educação Matemática) - Pontifícia Universidade Católica de São Paulo, 2003.

CHARTIER, Roger. *A história cultural.* Entre práticas e representações. Lisboa: Difel, 1990.

CHARTIER, Roger. Textos, impressão, leituras. HUNT, Lynn. *A nova história cultural*. São Paulo: Martins Fontes, p. 211-238, 1992.

CHOPPIN, Alain. O historiador e o livro escolar. *Revista História da Educação,* p.5-24, 2002.

DUARTE, Aparecida Rodrigues Silva. Omar Catunda e os debates sobre o ensino secundário de matemática na década de 1940. *Revista Iberoamericana de Educación Matemática*, n. 13, 101-114, 2008.

EVES, Howard. *Introdução à história da matemática.* Tradução: Hygino H. Domingues, Editora da Unicamp, 2004.

HAMLEY, Herbert Russell. Relational and functional thinking in mathematics. *National Council of Teachers of Mathematics.* Yearbook 9, 1934.

KRÜGER, Katja. *Functional thinking:* The history of a didactical principle. The legacy of Felix Klein, p. 35-53, 2019. DOI: https://doi.org/10.1007/978-3-319-99386-7_3

LONGEN, Adilson. *Livros didáticos de Algacyr Munhoz Maeder sob um olhar da educação matemática.* 2007. Tese (Doutorado em Educação) - Universidade Federal do Paraná, 2007.

MACIEL, Leandro Silvio Katzer Rezende. *Manoel Jairo Bezerra:* depoimentos em vida. **Zetetike**, *20*(1), 2012.

MIORIM, Maria Ângela. *Introdução à história da educação matemática.* Atual Editora, 1988.

MORAES, Carlos Roberto. Uma história da lógica no Brasil: A época dos pioneiros. *Revista Brasileira de História da Matemática.* v. 8, n. 15, 57-73, 2008.

OLIVEIRA FILHO, Francisco. *A matemática do colégio: livros didáticos e história de uma disciplina escolar*. 2013. Tese (Doutorado em Educação Matemática) - Universidade Anhanguera de São Paulo, 2013.

OLIVEIRA, Alexandre Souza. *A abordagem do conceito de função em livros didáticos ginasiais: Uma análise em tempos modernos (décadas de 1960 e 1970).* 2009. Dissertação (Mestrado em Educação Matemática). Universidade Bandeirantes de São Paulo, 2009.

PONTE, João Pedro. O conceito de função no currículo de Matemática. *Educação e Matemática.* n. 15, 3-9, 1990.

ROCHA, José Lourenço. *A Matemática do curso secundário na Reforma Francisco Campos.* 2001. Dissertação (Mestrado em Matemática) – Pontifícia Universidade Católica – RJ, 2001.

SILVA, Circe Mary Silva & BRUM, Ana Paula Rodrigues. Contribuições de Thales de Faria Mello Carvalho para a educação matemática no século XX. *Vidya*, v.42, n.2, 217-231, 2022. DOI: https://doi.org/10.37781/vidya.v42i2.4352

SILVA, Heloísa & BERNARDINO, C. L. À elite, uma terceira natureza de números na Coleção Matemática 2º Ciclo. *Zetetike*, *27*, e019017, 2019. DOI: https://doi.org/10.20396/zet.v27i0.8654316

TOBIES, Renate. Felix Klein—Mathematician, Academic Organizer, Educational Reformer. In: Weigand, HG., McCallum, W., Menghini, M., Neubrand, M., Schubring, G. (eds) The Legacy of Felix Klein. ICME-13, *Monographs*. Springer, Cham. p. 5-20, 2019. DOI: https://doi.org/10.1007/978-3-319-99386-7_1

VALENTE, Wagner Rodrigues. Educação Matemática e Política: a escolarização do conceito de função no Brasil. *Educação Matemática em Revista.* v. 12, n.1, p. 16-20, 2002.

VALENTE, Wagner Rodrigues. História da Educação Matemática: interrogações metodológicas. *Revemat: Revista Eletrônica de matemática.* v.2, n.1, p.28-49, 2007.

VALENTE, Wagner Rodrigues. Livro didático e Educação matemática: uma história inseparável. *Zetetike, 16*(2), 139–162, 2008.