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The Historical Movement of the Contributions of Grassmann's Extension Theory to Linear Algebra[1]

O Movimento Histórico das Contribuições da Teoria da Extensão de Grassmann para a Álgebra Linear

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ABSTRACT
Hermann Grassmann, a 19th-century German scholar, pioneered the foundations of Linear Algebra, a mathematical discipline crucial for diverse fields like Mathematics, Engineering, and Physics. This article endeavours to present a comprehensive biography of Grassmann, emphasizing two works documenting his contributions to this domain. Employing bibliographic research coupled with qualitative data analysis grounded in the logical-historical movement of the object, the study enables a historical, cultural, and social examination. Addressing the central question — what were Grassmann's contributions to the evolution of Linear Algebra, positioning him as its creator? — the inquiry concludes that Grassmann introduced notions integral to contemporary Linear Algebra, including n-dimensional vector spaces and linear transformations. Consequently, he garners recognition from many as the creator of Linear Algebra.

RESUMO
Hermann Grassmann foi um alemão do século XIX que estabeleceu os marcos iniciais para o surgimento da Álgebra Linear, um campo da Matemática que oferece recursos básicos para o desenvolvimento de várias subáreas da Matemática e de outras áreas do conhecimento, como Engenharias e Física, por exemplo. Assim, este artigo visa apresentar uma biografia deste autor, destacando duas de suas obras que registram seus feitos para este campo. Constitui-se de uma pesquisa bibliográfica com análise qualitativa dos dados pautada nos fundamentos do movimento lógico-histórico do objeto, possibilitando uma análise histórica, cultural e social, visando responder à seguinte questão-problema: quais foram as contribuições de Grassmann para o desenvolvimento da Álgebra Linear que o levou a ser considerado o criador desse campo de estudos? Conclui-se que Grassmann lançou as primeiras ideias sobre muitos conceitos que hoje constituem a Álgebra Linear, como espaços vetoriais n-dimensionais, transformação linear e outros, sendo, então, considerado por muitos, o criador da Álgebra Linear.

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Palavras-chave: História da Matemática; Álgebra Linear; Hermann Grassmann; Movimento lógico-histórico.

1 Introduction

Linear Algebra, as a field of mathematical study, has its roots in a historical development dating back approximately 4,000 years ago when the Babylonians discovered the process of solving systems of linear equations with two equations and two unknowns. Over time and with the emergence of new demands stemming from human activity, Linear Algebra evolved, benefiting from the collaboration of various notable scientists and mathematicians such as Cayley, Leibniz, Peano, among others.

At the peak of the the Industrial Revolution in the 19th century, a period marked by the prominence of scientific knowledge, the German Hermann Günther Grassmann (1809-1877), who worked as a theologian, mathematician, and high school teacher, published two books in 1844 and 1862. One of them titled Die Lineale Ausdehnungslehre, ein neuer Zweig der *Mathematik*[5] (GRASSMANN, 1878) and the other *Die Ausdehnungslehre*[6] (GRASSMANN, 1862), respectively. In these books, Grassmann formulated a mathematical theory previously unknown to researchers, employing a highly philosophical language, the fact that hindered its understanding by mathematicians of the time. Moreover, they contained a highly complex theory, leading to rejection by intellectuals. Nevertheless, Grassmann's theory laid the foundation for a theory that was about to emerge in the algebraic landscape, which will be discussed further herein.

Given Grassmann's significance in the context of the development and structuring of Linear Algebra as a mathematical field, this article aims to address the following inquiry: What were Grassmann's contributions to the advancement of Linear Algebra, and how did he become one of the pioneers in the creation of this field of study? It is worth highlighting that Grassmann played a fundamental

[5] The Theory of Extension, a new branch of Mathematic.
[6] The Theory of Extension.

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role in the structuring of algebraic theories, and subsequently, other researchers played important roles in this process.

In order to answering this inquiry, the specific objectives of this article are to present a biography of Grassmann's life, highlighting his main works, and to identify the concepts he created, comparing them with current algebraic notation. Therefore, the adopted research methodology is qualitative bibliographic, seeking elements from the logical-historical movement for the analysis and understanding of the facts, namely those related to the concepts of movement, fluency, interdependence, reality, totality, history, process, knowledge, mutability, logic, and thought (SOUSA, 2018).

It is noteworthy that research in the field of History of Mathematics plays a fundamental role in obtaining a deep and comprehensive understanding of the form and contents of mathematical science concepts, enriching Mathematical Education, and stimulating the development of theoretical thinking through the internal connections of concepts.

2 A biography of Hermann Günther Grassmann

According to Kopnin (1978), dialectical pairs constitute the totality of the object. Thus, the dialectical pair of life and work, permeated by Grassmann's logical-historical approach, leads to the totality of algebraic concepts presented in this text, constituting Linear Algebra. Through this interdependence, one can comprehend the relationship between the mutability and immutability of algebraic concepts, the connection between human thought and the reality of Grassmann's life, all embedded within the universal law, of movement, which encompasses both the logical and historical aspects of this author's life (SOUSA, 2018).

This way,

> Understanding the logical-historical movement of life entails comprehending that all knowledge contains anxieties, fears, afflictions, daring ventures, unexpected occurrences, new qualities, conflicts between the old and the new, between the past and the future. It is to understand that the totality of knowledge is the very movement of objective reality that is always yet to come (SOUSA, 2018, p. 45).

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With the aim of better organizing and understanding the facts that permeate the unity of Grassmann's life and work, the text will be divided into periods marking important stages of this movement, allowing for a comprehensive view of his achievements, and consequently, of Linear Algebra.

2.1 The student and professional life of Hermann Grassmann (1809 to 1843)

Hermann Günther Grassmann was born in 1809 in Stettin, at the time part of Prussia. He was a theologian, self-taught mathematician, high school teacher, and the third of twelve children of Justus Günther Grassmann and Johanne Luise Friederike Medenwald. His father, besides being a theologian, and a pietistic[7] and pastor, was also a mathematics teacher (DIEUDONNÉ, 1979).

Grassmann lived in a time when Germany did not exist as a unified nation, but was composed of 39 independent states. The most prominent and influential states within this confederation were the Austrian Empire and the Kingdom of Prussia. It is worth noting that, due to political and border changes resulting from wars and conflicts in the 20th century, Stettin is currently part of Poland. (ASSIS, 2018).

During this period, Enlightenment ideals were spreading throughout the world, influencing significant social and political changes. The Enlightenment thinkers advocated values such as democracy, economic liberalism, and rationalism, which were against to monarchical absolutism and the influence of the Church over the population. The Industrial Revolution also occurred during this time, transforming society and the economy, as well as human activity.

The political and intellectual landscape of that moment likely impacted Grassmann's ideas and may have influenced his view of Mathematics and his contributions to the field, as stated by Kopnin (1978, p. 53), "[...]The laws of the objective world become laws of thought as well, and all laws of thought are

[7] "Pietism emerged in seventeenth-century Protestant Germany. It accentuates personal faith in protest against the secularization of the Church. It emerged as a reaction to the "Thirty Years' War" in Germany and spread throughout Europe whenever religion divorced from personal experience. There were several immediate reasons for this movement, among them the scholastic hardening of Lutheranism in the face of its adversaries" (Rubem, 2023, n.p).

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represented in the objective world." With his innovative work in Algebra and Geometry, Grassmann left a mark of rigor on the development of Mathematics, contributing to the understanding and advancement of the field in a period of profound historical and intellectual changes.

Despite belonging to a family with a commendable level of education, Hermann did not stood out in the early years of secondary school, which today corresponds to elementary education. In his time, young individuals often demonstrated their aptitudes early on. Due to having attended reputable schools and received a good home education, his father believed that he could not possibly achieve any academic significance. However, gradually he began to make progress in school, eventually ranking second in the final exam of what nowadays known as high school. (DIEUDONNÉ, 1979).

Determined to study theology, driven by his desire to become a Lutheran minister, in 1827 he enrolled at the University of Berlin, the oldest institution in the city and one of the most privileged in Germany and Europe. There is no information indicating that Grassmann had any academic background in Mathematics; however, it was mathematics that led him back to Stettin after completing his studies in Berlin in 1830, where he became a teacher at secondary schools and embarked on independent mathematical research (DIEUDONNÉ, 1979).

Upon returning to Berlin in 1831 to take examinations for an academic career, he did not perform well, and he was only permitted to teach Mathematics at lower levels. In 1832, he was appointed as an assistant professor at the Gymnasium of Stettin, and during this period, he made his first mathematical discoveries, which would later be developed and published.

Due to the nature of his professional license, in 1835, Grassmann was appointed to teach Mathematics, Physics, German, Latin, and Religious Studies at the Otto Schule in Stettin, a primary school. Then, in 1839, he passed his theology exams, which allowed him to teach at the secondary level.

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> It is important here to put some emphasis on what were the duties of a Gymnasium professor in Germany at that time, and in fact all through Grassmann's lifetime: he was supposed to lecture on all subjects, from religion to biology through latin, mathematics, physics and chemistry, and at all levels, at a rate of 18 to 30 hours a week (DIEUDONNÉ, 1979, p. 2).

Grassmann aspired to teach at the university level so that he could dedicate more time to his research. In the preface of his Theory of Extension of 1844, one can discern his restlessness due to his profession not allowing him to pursue scientific research, as he desired:

> On one particular ground I hope to find indulgence, that the time for my research was extremely paltry and measured out piecemeal by virtue of my official position. Also my position offered me no opportunity, through reports on the domain of this science, or even related matter, to obtain that vivid freshness which must as an enlivening breath inspire the whole if it is to appear as a living limb of the organism of science. (GRASSMANN, 1995, p. 17).

In 1840, Grassmann took admission exams for the Stettin Institute, which were significant for him despite the contradictions he experienced there. He had to present an essay on the theory of tides, for which he utilized the basic theory of Celestial Mechanics by Laplace and Analytical Mechanics by Lagrange, producing his essay Theory of Flow and Reflux, which introduced, for the first time, an analysis based on vectors (DIEUDONNÉ, 1979).

Dieudonné (1979) asserts that, although Grassmann's essay was accepted by the examiners, they did not deem the innovations introduced therein important, as his explanation was considered too abstract and philosophical. It is evident how the form of concepts was important for the mathematicians of the time, and that, lacking the academically accepted form, the content of the concepts was also disregarded. However, realizing that his theory had broad applicability, Grassmann decided to dedicate himself to the development of his ideas on vector spaces, including vector addition and subtraction, vector differentiation, and the theory of vector functions.

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In 1843, Grassmann returned to Berlin and became interested in Geometry due to exams he took to improve his teaching position. According to Dieudonné (1979), Grassmann arrived at the understanding of vectors on his own, as he had not studied Mathematics beyond the high school curriculum. This reveals that Grassmann possessed a theoretical mindset and was able to establish both the internal and external connections of mathematical concepts.

Grassmann aspired to become a university professor, believing that this would facilitate the promotion of his ideas and, consequently, the dissemination of his new theory. Sousa (2018) asserts, "Human thought seeks ways to enable the continuous transformation of reality through its physical and intellectual labor [...]". He repeatedly submitted his works for evaluation by experts in the hope of obtaining a position as a university lecturer. However, they failed to comprehend him, and consequently, rejected him.

2.2 From the publication of his first work to its rewriting (1844 to 1861)

After all attempts to become a university professor were denied, Grassmann turned his attention to writing his work *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik* (GRASSMANN, 1878), which was published in 1844. He created what is now called exterior algebra[8]. However, his work did not receive much receptivity and was criticized by "[...] He was mostly reproached for lack of clarity in his presentation, especially regarding the long philosophical introduction that prevented most mathematicians from reading any further." (DORIER, 2000, p. 18).

Grassmann seemed to be aware of the potential rejection his work might encounter, as he wrote:

> In fact, when presenting a new science, so that its position and meaning can be duly recognized, it is essential to show at once its application and its relationship with related objects. The introduction should also serve this purpose. In the nature of things,

[8] "Exterior Algebra, from a purely algebraic standpoint, is the study of alternating multilinear applications and their ramifications. From a geometric perspective, it deals with p-dimensional vectors, originally conceived by H. Grassmann" (Lima, 2017, translation ours, n.p).

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> this is more philosophical in nature, and when I separated it from the context of the entire work, I did so so as not to immediately scare mathematicians with the philosophical form. Because there is still a certain rejection among mathematicians, and partly not mistakenly, towards philosophical discussions of mathematical and physical subjects (GRASSMANN, 1995, p. 15-16).

It is noted that Grassmann justified the lack of success of his work due to its presentation, which employed an unconventional language for the time, being rather philosophical and with little mathematical writing. This reflects the interdependence between the form and content of mathematical concepts, both essential for the development of this science.

In 1846, Möbius suggested, according to Dieudonné (1979), that Grassmann compete for the *Fürstlich Jablonowskischen Gesellschaft* prize (awarded to those who have had great relevance in the exact sciences) from the Leipzig Academy. He then submitted his book titled *Die Geometrische Analyze geknüpft und die von Leibniz Characteristik*, where he developed Leibniz's ideas on establishing a calculus directly applicable to geometric situations (OTTE, 1989). In this study, according to Dieudonné (1979), Grassmann introduced the notion of scalar product between two vectors belonging to n-dimensional spaces. However, Möbius, who was one of the judges, criticized the way Grassmann introduced abstract ideas without providing the reader with an intuitive hook to hang them on (SERVIDONI, 2006).

In May 1847, Grassmann was granted the title of *Oberlehrer* senior teacher) at *Friedrich Wilhelm Schule*, and in the same month, he wrote to the Prussian Ministry of Education requesting that his name be included on a list of those to be considered suitable for university positions. The Ministry of Education sought Kummer's opinion on Grassmann, who, through a report, dashed any hope Grassmann had of obtaining a university position, stating that the request was inadequately expressed (DOURADO *et al.*, 2021).

In early 1849, he married Therese Knappe, with whom he had eleven children, nine of whom reached adulthood (EVES, 2004). In March 1852, Grassmann's father, Justus, passed away from heart failure. Later that year, Grassmann was appointed to fill his late father's former position in *Stettin Gymnasium* (DIEUDONNÉ, 1979).

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Having failed to gain recognition for his mathematics, Grassmann turned to one of his other favorite subjects, the study of Sanskrit and Gothic. Throughout his life, he gained more recognition for his language studies than for mathematics; even proving that Germanic was older in a phonological pattern than Sanskrit, establishing it as the oldest language in Indo-European linguistics. Indeed, Grassmann's remarkable versatility is evident in his dedication to numerous fields, displaying the breadth of his scientific knowledge.

In 1854, he redirected his focus to mathematics, his great passion, and his theory of extension. He was determined to rewrite his work fully, hoping to get the recognition he deserved. Simultaneously, by the year 1861, he published seventeen scientific papers that he developed for the fields of physics and mathematics. His intellectual breadth, as pointed out by Servidoni (2006), allowed him to navigate forcefully through various areas of knowledge.

2.3 From the second version of the Theory of Extension to his death (1862 to 1877)

In 1862, a new version of his 1844 work, titled *Die Ausdehnungslehre,* was published (GRASSMANN, 1862). In this version, Grassmann defines mathematical concepts and immediately proceeds to present the applications of these concepts, thus making his ideas clearer. The alteration in the structure of the text is notable, particularly the complete omission of the philosophical introduction, as well as a methodological restructuring of the text in favor of a format closer to propositional organization, characteristic of Euclidean style. It becomes clear that, in this movement, through a process of mutability, Grassmann would be seeking to adapt his proposal to receive a positive reception, which, once again, would not be successful, at least in the 1860s (LINDER; SCHUBRING, 2022). Dorier (2000) points out that, even though it no longer contained the heavy philosophical language of the first version, the reading was still dense and required meticulous study of the entire theory. This fact led the mathematicians of the time to ignore his work once again, denying him a voice within the scientific community.

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Disappointed at his inability to convince mathematicians of the importance of his work, Grassmann turned once again to research in linguistics, where here, he was honored for his contributions by being elected to the *American Oriental Society* and awarded an honorary degree by the University of Tübingen in Germany (DOURADO *et al.*, 2021).

In 1872, Grassmann resumed his mathematical studies upon learning that mathematicians were investigating and applying his works and theories. According to his son, Justus Grassmann, two days before his passing, Grassmann received the news that his theory would be disseminated through lectures given by a professor in Dresden (GRASSMANN, 2011).

Grassmann passed away on September 26, 1877, due to cardiac problems following a period of greatly compromised health, as indicated by Servidoni (2006). Dieudonné (1979, p. 1) speaking about Grassmann, asserts that:

> In the whole gallery of prominent mathematicians who, since the time of the Greeks, have left their mark on science, Hermann Grassmann certainly stands out as the most exceptional in many respects [...]. When compared with other mathematicians, his career is an uninterrupted succession of oddities: unusual were his studies; unusual his mathematical style; highly unusual his own belated realization of his powers as a mathematician; unusual and unfortunate the total lack of understanding of his ideas, not only during his lifetime but long after his death; deplorable the neglect which compelled him to remain all his life professor in a high-school ("Gymnasiallehrer") when far lesser men occupied University positions.

Grassmann's theory, which aligns with what Kopnin (1978, p. 196) asserts to be a scientific theory, namely "a system, a set of judgments unified by a single principle". Although not widely accepted in his time, contained the foundations of a theory that was about to emerge in the field of Algebra, including concepts such as vector space, linear dependence, basis, dimension, and linear transformation. Due to these valuable contributions, He is widely acknowledged as the progenitor of Linear Algebra (ASSIS, 2018).

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According to Whitehead (1898), it is believed that Grassmann, while writing his second version of the Theory of Extension, was not even aware of Cayley's classic memoir on matrices. According to Dorier (1995), Grassmann formulated his theory independently of the rest of mathematics, relying solely on elementary rules of mathematical theoretical thought.

Grassmann's theories only gained recognition after Giuseppe Peano published a condensed version of his own interpretation of Grassmann's work, which he titled *Geometric Calculus – according to the Ausdehnungslehre of H. Grassmann* (PEANO, 2000).

Peano, being Italian, was the most renowned mathematician of his time in his country. Despite his influence, he did not gain notoriety in his work on Geometric Calculus, likely, because he presented Grassmann's ideas rather than his own.

3 Grassmann's contributions through Peano's perspective.

In your Geometric Calculus (PEANO, 2000), Peano conducted a reinterpretation of Grassmann's Theory of Extension, transforming philosophical language into a more mathematical one, thereby gaining greater acceptance within the scientific community of the time. Additionally, he rectified certain misconceptions made by the author and expanded upon some already established concepts. Nevertheless, Peano makes it clear in his work that such production deals with concepts already conceived by Grassmann, and that his objective is not only to give more visibility to the author but also to make his theories known and accepted by the mathematical society of his time (ASSIS, 2018), as he himself asserts:

> The intent of the present book is the direct explanation, in a form accessible to anyone cognizant of the fundamentals of geometry and algebra, of a geometric calculus, based on some notations contained in the Ausdehnungslehre, and in the development of its principal consequences (PEANO, 2000, p. 9).

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Peano, in developing this work, had to comprehend the process of becoming inherent in the mathematical concepts presented by Grassmann, apprehending the Grassmannian concepts through the attainment of the universal, the essential, and the necessary to the concept (KOPNIN, 1978).

The aim of Peano in this book was the same as Grassmann's, the development of a geometric calculus. Thus, the book largely consists of calculations involving points, lines, planes, and three-dimensional figures. According to Dorier (1995), Peano recovers the main concepts contained in the Theory of Extension (1862 version), presenting them in a well-constructed and summarized manner, with applications placed at the end of each chapter, similar to Grassmann. In the terms of Kopnin (1978), Peano represented the same object in a different manner, with a different purpose, to serve another function in the movement of thought leading to objective truth and contributing to the process of reproducing the object through thought.

In the first chapter of his work, after defining the concepts of volume, surface, and oriented segments, Peano establishes operations concerning four different types of geometric objects: points, segments, surfaces, and volumes. In this process, he extracts broader properties that are not linked to a specific type of geometric object. This demonstrates that it is possible to carry out operations with these geometric formations, irrespective of the particular algebraic objects involved. (PEANO, 2000). As he states in the preface of the book, "Geometric calculation consists of a system of operations applicable to geometric beings, analogous to those that algebra performs on numbers [...]". (DORIER, 1990, p. 64). This fact reveals the interdependence of the way of thinking about objects in Linear Algebra with other algebraic and geometric objects.

In Chapter Two, he demonstrates that in a space, if three vectors are such that their product, expressed as a determinant, is non-zero, then all vectors can be written as a linear combination of these three vectors (DORIER, 1995), introducing the notion of vectors generating a space.

In the ninth and final chapter, Peano presents a definition of what he termed a linear system, which we now know as a vector space. This is the first axiomatic

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definition of a real vector space and was very close to the current definition, which is the synthesis of a historical process of development. There are also other definitions with this characteristic, namely, linear independence, basis, coordinates, and dimension (considering the possibility of infinite dimension). In addition, there is the definition of the dimension of a linear system as the maximum number of linearly independent quantities in a system. In the remaining chapters, Peano presents all the remaining Grassmannian theory, which originated Grassmann's Algebra.

The significance of Peano's work in disseminating Grassmann's ideas is evident. Despite Grassmann's work being underrated in his time, his contributions have had a significant impact on the development of Linear Algebra.

The contributions regarding the formation of the aforementioned concepts will be discussed in the following sections, drawing parallels with what is now formalized in textbooks on Linear Algebra. This will demonstrate that in Grassmann's work, the essence of the concept was already present. However, the focus is not on the linear aspect of history, but rather on the mutability of the history of concepts, which contains a reality in constant flux and cannot be divided or fractionated but must be viewed as a whole. For "The essence of this reality lies in fluency, movement, transformation, and not in the fragmentation of human thought itself, which contains interdependence, a fundamental characteristic of the movement of thought" (SOUSA, 2018, p. 47, our translation).

3.1 Vectors

According to Kleiner (2007), the earliest notions of vectors do not date back to recent centuries, nor are they related to Mathematics, but are linked to Physics and trace back to thinkers of Ancient Greece, who interpreted vectors with a "very strong geometric and intuitive appeal to aid in the analysis of physical problems" (TÁBOAS, 2010, p. 2, our translation). According to Kleiner (2007) and Táboas (2010), ancient Greeks employed the concept of vectors and vector addition (nowadays known as the parallelogram rule) in the context of physical quantities such as force and velocity. These are concepts emerging from human praxis.

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In Mathematics, the notion of vectors originated from the geometric representation of complex numbers, independently introduced by various authors in the late 18th and early 19th centuries. This representation was limited to points or oriented line segments on a plane. Hamilton, in his astronomical research, needing to work with rotations in the plane, found in the graphical representation of complex numbers the appropriate mathematical theory. However, he needed to relate this theory to the theory of vectors. Thus, in 1835, he algebraically defined complex numbers as ordered pairs of real numbers, possessing the usual operations of addition, multiplication, and scalar multiplication, as we have today (KLEINER, 2007). In this way, Hamilton associated complex numbers with vector representation, marking the first notions of mathematical vectors and already demonstrating the interdependence of mathematical representations of concepts.

During the same period as Hamilton, Grassmann, in the second version of his work, represented a vector as a straight-line segment with specific length and direction, introducing a geometric representation for this concept, thus illustrating the mutability of conceptual notions. He defines the addition of two vectors according to the rule that we now commonly accept, and subtraction of vectors is performed as the addition of the negative, which is the vector that has the same length and opposite direction (MELCHIADES DA SILVA; FRANT; OLIVEIRA, 2022).

Peano, in his interpretation of Grassmann's work, defined a vector as follows: “Every formation of the first species of the form B - A is called a vector. The points A and B are called respectively the origin and term of the vector.” (PEANO, 2000, p. 30). For the author, formation of the first kind equates to a line; consequently, a formation of the form $B - A$ becomes a limitation of this line, which would be, in modern language, an oriented line segment leading to the current conceptualization of a vector.

Considering the contemporary approach to the concept of vectors, Boldrini et al. (1986) state that vectors in the plane are oriented line segments with the initial point at the origin, uniquely determined by their terminal point, since the starting point is fixed at the origin. Algebraically, for each point in the plane, a

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unique vector is associated, and given a vector, we associate a unique point in the plane, which is its terminal point. Thus, a correspondence called biunivocal is established between points in the plane and vectors.

Extending the definition of vectors to three-dimensional space, Boldrini et al. (1986, p. 101) states that in this space there is also "a coordinate system given by three oriented lines, pairwise perpendicular, and, once a unit length is fixed, each point P in space will be indicated as a triple of real numbers (x, y, z), which gives its coordinates."

It is noted that Grassmann, as expressed by Peano (2000), already conceived the core of the vector concept, and that later authors, through the historical development of the concept, refined their writing, leading to a synthesis, as developed by Boldrini et al. (1986).

3.2 Vector Space

Regarding spaces with more than two dimensions, Eves (2004) asserts that since the time Euclid published his famous work "Elements of Euclid" around 300 B.C., the Greeks already knew about geometric solids and had geometry in three dimensions. Eves (2004) reports that in Book XIII of Euclid's "Elements," the author attributes the creation of some regular solids to the Pythagoreans, suggesting the emergence of these solids centuries before the 3rd century B.C.

With the advancement of science, it became necessary to develop an algebraic theory for a space with more than two dimensions, an expansion of the existing scientific theory up to that point. After several unsuccessful attempts by various scholars to create a three-dimensional space, in 1843, Hamilton introduced quaternions, establishing a four-dimensional set; however, he lacked a solid theory for a vector space of more than two dimensions. In this context, Grassmann introduced his Theory of Extension in 1844 and 1862, proposing an extremely abstract algebra for mathematicians of the time, yet containing important traces of n −dimensional vector analysis required by scientists. It represents the evolution of mathematical theory demanded by scholars, the fluidity of concepts.

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In 1888, while elucidating Grassmann's ideas, Peano introduced his linear systems, which we now know, in modern terms, as vector spaces. He stated that a linear system is a system of entities that satisfy the conditions 1, 2, 3, and 4 below:

> Let there exist a system of entities for which are given the following definitions:
> 1. The equivalence of two entities a and b of the system is defined, that is, a proposition, indicated by $a = b$, is defined, which expresses a condition between two entities of the system, satisfied by certain pairs of entities, and not by others, and which satisfies the logical equations:
> $$(a = b) = (b = a), (a = b) \cap (b = c) < (a = c)$$
> 2. The sum of two entities a and b is defined, that is to say an entity, indicated by $a + b$, is defined that also belongs to the system given, and which satisfies the conditions
> $$(a = b) < (a + c = b + c), a + b = b + a, a + (b + c) = (a + b) + c,$$
> the common value of the two members of the last equivalence being indicated by $a + b + c$.
> 3. If a is an entity of the system, and m is a positive integer, by the expression ma we will mean the sum of m entities equal to a. It is easy to recognize that, if $a, b, \ldots$ are entities of the system, and $m, n, \ldots$ positive integers,
> $$(a = b) < (m_a = m_b); m(a + b) = m_a + m_b;$$
> $$(m + n)a = m_a + n_a; m(na) = (mn)a; 1a = a.$$
> We will suppose that a meaning is attributed to the expression rna, whatever may be the real number m, in such a way that the preceding equations are also satisfied. The entity ma is called the product of the (real) number m with the entity a.
> 4. Finally we will suppose there exists an entity of the system, which we will call the null entity, and that we will indicate by 0, such that whatever may be the entity a, the product of the number 0 with the entity a always gives the entity 0, that is $0a = 0$. If to the expression $a - b$ one attributes the meaning $a + (-1)b$, one deduces $a - a = 0, a + 0 = a$ (PEANO, 1888, p. 119-120).

Dorier (2000) clarifies that while Grassmann deduced the fundamental properties of vector spaces from the definition of operations on coordinates, Peano was more precise and described an axiomatic structure from which they emerged. Peano also refined the formulation by removing some redundancies and providing greater clarity to the concepts of zero and opposite elements. Thus, Peano went a step further than Grassmann, contributing to the improvement and development of the concept of vector space, leading to a mutation of ideas through established judgment, towards a relatively finished idea (KOPNIN, 1978), following his studies of the Theory of Extension.

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With the contribution of other scholars, the concept continued to develop, transform, and its expression improved until reaching the synthesis we have today, as can be expressed according to Boldrini et al. (1986), defining a vector space as a non-empty set V that possesses addition and scalar multiplication operations, given, respectively, by

$$
\begin{array}{ll}
+: V \times V \to V & \cdot: R \times V \to V \\
(u, v) \mapsto u + v & (a, u) \mapsto au
\end{array}
$$

such that, for any $u, v, w \in V$ and $a, b \in R$, the following properties are satisfied:

$i (u + v) + w = u + (v + w)$
$ii u + v = v + u$
iiiThere exists $0 \in V$ such that $u + 0 = u$.(Here,0it is called the zero vector.)
ivThere exists $-u \in V$ such that $u + (-u) = 0$
$v a(u + v) = au + av$
$vi (a + b)v = av + bv$
$vii (ab)v = a(bv)$
$viii 1u = u$ (BOLDRINI *et al.*, 1986, p. 103, our translation).

With this definition, the richness of Peano's contributions to the axiomatization of Mathematics is evident, as through this axiomatization, it was possible to establish clear and concise syntheses of mathematical concepts such as that of a vector space.

3.3 Linear Dependence and Independence

Grassmann, according to Táboas (2010), did not originally conceive the concepts of linear dependence and independence. However, the author states that Grassmann, in his Theory of Extension, presented significant aspects for the organization and formalization of these concepts in axiomatic terms, placing them in a broader context within Mathematics.

Grassmann approached the definition of linear dependence by stating: "We say that an elementary quantity of the first degree is dependent on other elementary quantities when it can be represented by a linear combination of these latter" (MELCHIADES DA SILVA; FRANT; OLIVEIRA, 2022, p. 20-21, our translation).

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Grassmann also referred to the concept of dependence as one species being dependent on another species, when a vector could be written as a linear combination of other vectors (MELCHIADES DA SILVA; FRANT; OLIVEIRA, 2022), exactly as we have it today.

We also find the notion of linear dependence expressed in the following terms: "One kind of change *(Aenderungsweise)* is dependent on another when the vectors of the first can be represented as a sum of vectors of the second [...]" (MELCHIADES DA SILVA; FRANT; OLIVEIRA, 2022, p. 20, our translation). However, according to the author, this statement is not sufficient to decide when the given vectors are linearly independent or, in Grassmann's terms, independent of each other, as there are still inherent contradictions in it.

Caire (2020) also highlights a notion of linear dependence presented in the context of what Grassmann referred to as magnitudes:

> [Grassmann] defined the concept of dependence, where an elementary magnitude of the first order was represented as a multiple sum of independent elementary magnitudes of the first order, and these independent magnitudes could not be represented as a multiple or sum of the remaining independent magnitudes (CAIRE, 2020, p. 80, our translation).

Through Peano, we see an improvement in the mathematical terms expressed by Grassmann, with the elimination of conceptual contradictions. In Chapter IX of his "Geometric Calculus", Peano provides the following definition of linearly dependent and independent.

> Several entities $a_1, a_2, \dots a_n$of a linear system are called mutually dependent if one can determine n numbers $m_1, m_2, \dots m_n$, not all zero, for which there results exist a relation
> $$m_1 a_1 + \cdots + m_n a_n = 0.$$
> In this case anyone of the entities whose coefficient is not zero can be expressed as a linear homogeneous function of the rest. If the entities $a_1 \dots a_n$ are mutually independent, and if between them there is a relation$m_1 a_1 + \cdots + m_n a_n$, one deduces $m_1 = 0, \dots, m_n = 0$. If $AB \dots$ are formations of first species in space, the equations $AB = 0, ABC = O, ABCD = 0$ express the dependence of 2 or 3 or 4 formations. 5 formations of the first species in space are always mutually dependent (PEANO, 1888, p.120).

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Comparing Peano's language with that used in contemporary textbooks, it is noted that the term "mutually" is equivalent to the term "linearly." Boldrini et al. (1986) state that it is of utmost importance to know whether a vector is or is not a linear combination of others, and they provide a synthesis of the entire historical development of the concepts of linearly dependent and linearly independent, expressed as follows:

> Let V be a vector space and $v_1, \dots, v_n \in V$. We say that the set $\{v_1, \dots, v_n\}$ is *linearly independent* (LI), or that the vectors $\{v_1, \dots, v_n\}$ are LI, if the equation $a_1 v_1 + \cdots = a_n v_n = 0$ implies that $a_1 = a_2 = \cdots = a_n = 0$. In the case where there exists some $a_i \neq 0$, we say that $\{v_1, \dots, v_n\}$ is *linearly dependent* (LD), or that the vectors $v_1, \dots, v_n$ are LD (BOLDRINI et al., 1986, p. 114, emphasis added).

The author further presents a result, in the form of a theorem, to clarify and classify linearly dependent vectors: "$\{v_1, \dots, v_n\}$ It is LD if, and only if, one of these vectors is a linear combination of the others" (Boldrini *et al.*, 1986, p. 114). The resemblance of the current notation to the notation introduced by Peano almost a hundred years before the work of Boldrini et al. (1986) is noticeable, reaffirming that the evolution of this concept manifests through the movement of the history of its formation and development (SOUSA, 2018).

3.4 Base and dimension

Crowe (1967), analyzing the works of Grassmann, brings the following result, which he says is a typical assertion of Grassmann: "Every vector of a system of the m^{th} order can be expressed as the sun of m given independet manners of change of the system. This expression is unique (CROWE, 1967, p. 69)." (Crowe, 1967, p. 69). In this passage, the concept of the dimension of a vector space is introduced when he speaks of a system of order m, and when he says "the sum of m vectors belonging to the given m independent ways of changing the system," he is referring to a linear combination of vectors that are linearly independent. Combining these two parts, it can be affirmed that Grassmann was referring to the basis of a vector space, which is a set of linearly independent vectors that span

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the entire space. Moreover, being spanned implies being written as a linear combination of some given vectors; consequently, these given vectors are the elements of the basis, whose quantity represents the dimension of the space.

Since the initial representations of the concepts of basis and dimension, their interdependence with other concepts such as linear combination and linear dependence is evident, demonstrating that the concepts are established within a conceptual system that requires theoretical thinking and specific mathematical forms of thinking that relate the concepts in order to develop new concepts.

From a contemporary perspective, this result can be rewritten as: Every vector in a vector space of dimension m can be written as a linear combination of m linearly independent vectors. This result resembles what Boldrini et al. (1986, p. 120, our translation) present as a theorem, which states: "Given a basis $\beta = \{v_1, v_2, \dots, v_n\}$ of V, each vector in V is uniquely written as a linear combination of $v_1, v_2, \dots, v_n$."

Peano (2000, p. 120), refining Grassmann's language, in defining dimension, writes that "The number of dimensions of a linear system is the maximum number of mutually independent entities of the system that one can take." To make his definition clearer, he provides an example:

> For example, the formations of the first species on a right-line, or in the plane, or in space, form linear systems of 2, 3, and 4 dimensions, respectively; the vectors in the plane or in space form systems of 2 and 3 dimensions; the formations of the second species in space form a system of 6 dimensions. The real numbers form a linear system of one dimension; the imaginary numbers, or ordinary complexes, form a system of two dimensions. A linear system can also have infinite dimensions (PEANO, 1888, p.120).

It is evident that Peano refers to the existence of vector spaces of infinite dimension; consequently, the notion of basis is implicit in this concept through its internal connections.

Formalizing all these concepts, Boldrini et al. (1986) writes that a set $\{v_1, \dots, v_n\}$ of vectors in a vector space V will be a basis of V if they are linearly independent and span the entire space. Moreover, "Any basis of a vector space

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always has the same number of elements. This number is called the *dimension* of V, denoted $dimV$." (BOLDRINI et al., 1986, p. 119, author's emphasis). Thus, we have a synthesis of the concepts initially presented by Grassmann and refined by Peano and other authors who, through the development of history, contributed to the improvement of mathematical notation.

3.5 Linear Transformation

In the 1862 version of the Theory of Extension, Grassmann added a part that was not present in the 1844 version and named it *Funktionenlehre* (Theory of Functions), which, according to Liesen (2011), it is the most splendid addition made by Grassmann. It is in this part that we find a section entitled "*Ganze Funktionen ersten Grades und Darstellung derfelben als Quotienten"* (Complete functions of the first degree and their representation as quotients), where Grassmann works with the mathematical object called "*Bruch*" or "*Quotient*" (Fraction or Quotient). In modern terminology, Grassmann's "fraction" can be referred to as a linear transformation of a finite-dimensional vector space (LIESEN, 2011).

Peano (2000), in representing history through logic via his interpretation of Grassmann, rewrites the definition of linear transformation as follows:

> An operation$R$, to be carried out on every entity a of a linear system$A$, is called distributive, if the result of the operation R on the entity a, which we will indicate by Ra, is also an entity of a linear system, and the identities
> $$R(a+a') = Ra + Ra',\ R(ma) = m(Ra),$$
> are verified, where a and a' are any entities whatever of the system A, and m is any real number. The entityRa, that is the result of the distributive operation R on the entity a, is called a distributive function of a. A distributive operation is also called a linear transformation, or transformation without any qualifier (PEANO, 1888, p. 122).

Through a temporal movement, bringing the concept of linear transformation into the present day, Boldrini et al. (1986, p. 144, author's emphasis) write:

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> Let V and W be two vector spaces. A *linear transformation* (linear mapping) is a function from V to$W$, denoted as $F: V \to W$, that satisfies the following conditions:
> i) For any vectors u and v in$V$, $F(u+v) = F(u) + F(v)$,
> iiFor any scalar $k \in R$and vector $\in V$, $F(kv) = kF(v)$.

It is notable that both works define linear transformation in the same manner, despite the different symbols used, indicating that in Peano's time the concept was already forming and that, with history, there was a mutation in mathematical symbolic writing.

4. Final Remarks

The historical development of the concepts presented reveals, in itself, their process of formation permeated by scientific certainties and uncertainties that evolved towards conceptual unity, namely, the synthesis of each concept expressed in current textbooks. The entirety of Linear Algebra extends beyond this historical process and occurs through the establishment of logic, influenced by history, in the formation of theoretical thought that evolves within the human mind through the learning process.

The similarity between the concepts developed by Grassmann, rewritten by Peano, and those used today in Linear Algebra reveals that the internal and external connections of the concepts have evolved with history, reflecting the movement of human thought towards conceptual synthesis.

Peano served as a mediator between the complex work of Grassmann and the mathematical community, reintroducing Grassmann's theory in a more accessible language that was consequently accepted by mathematicians and scholars of the time.

His accomplishments transcend his time, and with the improvements made by great figures such as Élie Cartan, Hankel, Whitehead, and Klein, Linear Algebra developed, structured, and reached its synthesis, which is found in current textbooks. It was the transformation of mathematical reality through the intellectual work of great minds.

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Through the concepts introduced by Grassmann, mathematicians developed new concepts and theories, advancing the process of knowledge through the mathematical generalization of Grassmann's theory. An example of this is the theory of infinite-dimensional vector spaces, which was propelled by a consideration made by Peano regarding the set of polynomial functions of a real variable. However, Banach only completed the axiomatic synthesis of this concept in 1920.

Other theories also developed from Grassmann's ideas, such as the axiomatization of a normed vector space created by Hilbert, the definition of a basis of a field extension developed by Dedekind, and the axiomatization of a complete normed vector space created by Banach.

The importance of Grassmann's Theory of Extension for the development of Mathematics is evident, requiring further research into this dense work that leads to an understanding of facts that permeate the teaching-learning processes of Linear Algebra today, whether in the field of History of Mathematics, Mathematics Education, or other related areas.

El Movimiento Histórico de las Aportaciones de la Teoría de Extensión de Grassmann al Álgebra Lineal

RESUMEN
Hermann Grassmann fue un alemán del siglo XIX que estableció los hitos iniciales para el surgimiento del Álgebra Lineal, un área de las Matemáticas que ofrece recursos básicos para el desarrollo de diversas subáreas de las Matemáticas y de otros ámbitos del conocimiento, como la Ingeniería y la Física, por ejemplo. Por ello, este artículo tiene por objeto presentar una biografía de este autor, destacando dos de sus obras que recogen sus logros en esta área. Consiste en una investigación bibliográfica con análisis cualitativo de datos basado en los fundamentos del movimiento lógico-histórico del objeto, posibilitando un análisis histórico, cultural y social, con el objetivo de responder a la siguiente pregunta problema: ¿cuáles fueron las contribuciones de Grassmann al desarrollo del Álgebra Lineal que lo llevaron a ser considerado el creador de este campo de estudio? La conclusión es que Grassmann lanzó las primeras ideas sobre muchos de los conceptos que hoy componen el Álgebra Lineal, como los espacios vectoriales n-dimensionales, la transformación lineal y otros, por lo que muchos lo consideran el creador del Álgebra Lineal.

Palabras clave: Historia de las Matemáticas; Álgebra Lineal; Hermann Grassmann; Movimiento Lógico-Histórico.

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