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Is Aristotle's cosmos hyperbolic?

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Summary: Aristotle's refusal of the actual infinite, in any form, leads him to conceive a universe finite in magnitude, containing a finite multiplicity of things. His strict "immanentism" implies that not only physics but mathematics too must be done in this real universe, without concessions to imagination: in Aristotle's mathematics there are no sets of actually infinite elements, nor lines of actually infinite length. Even worse, there are not even lines of finite length potentially infinitely extendible, no curves going to the infinite. This notwithstanding, Aristotle explicitly says that his restricted way of understanding the infinite is not a problem for mathematicians. Fortunately, he goes further than the mere statement explaining why mathematicians can do without infinite sets, infinite lines and infinitely extensible ones.

Keywords: Aristotle. Infinite. Philosophy of mathematics.

O Universo de Aristóteles é hiperbólico?

Resumo: A refutação do infinito, em toda a sua amplitude, leva Aristóteles a conceber um universo finito em grandeza e detentor de multiplicidade finita de objetos. A sua concepção estritamente "imanentista" implica que um tal universo finito deva abarcar não só a física, mas também a matemática, sem qualquer concessão à imaginação: na matemática de Aristóteles não há, então, conjuntos de elementos infinitos, nem linhas de comprimento infinito. Mais ainda: nem sequer há linhas infinitamente estendíveis ou curvas que rumam ao infinito. Não obstante isso, Aristóteles afirma que a sua visão restritiva do infinito não é exatamente um problema para os matemáticos, e explica de que modo podem abrir mão de conjuntos infinitos e de linhas infinitamente estendidas.

Palavras-chave: Aristóteles. Infinito. Filosofia da matemática.

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L'Universo di Aristotele è iperbolico?

Riassunto: Il rifiuto dell'infinito attuale, in ogni sua forma, porta Aristotele a concepire un universo finito in grandezza, contenente una molteplicità finita di oggetti. La sua concezione strettamente "immanentista" implica inoltre che in un tale universo finito debba essere contenuta non solo la fisica ma anche la matematica, senza alcuna concessione all'immaginazione: nella matematica di Aristotele non ci sono dunque insiemi di infiniti elementi, né linee di lunghezza infinita. Ma non solo: non ci sono nemmeno linee infinitamente estendibili, né curve che vanno all'infinito. Ciò nonostante, Aristotele afferma che la sua visione così restrittiva dell'infinito non sia un problema per i matematici, e spiega in che modo i matematici possano fare a meno di insiemi infiniti e di linee infinitamente estendibili.

Parole chiave: Aristotele. Infinito. Filosofia della matemática.

Aristotle's refusal of the actual infinite, in any form, leads him to conceive a universe finite in magnitude, containing a finite multiplicity of things. His strict "immanentism" implies that not only physics but mathematics too must be done in this real universe, without concessions to imagination: in Aristotle's mathematics there are no sets of actually infinite elements, nor lines of actually infinite length. Even worse, there are not even lines of finite length potentially infinitely extendible, no curves going to the infinite.

This notwithstanding, Aristotle explicitly says that his restricted way of understanding the infinite is not a problem for mathematicians. Fortunately, he goes further than the mere statement explaining why mathematicians can do without infinite sets, infinite lines and infinitely extensible ones. Unfortunately, his explanation is cryptic, and not so easy to follow. The aim of this paper is to expand upon Aristotle's explanation, analysing in particular the iterative procedure of *converse increasing* (ἀντεστραμμένη πρόσθεσις), which he introduces in order to solve the problem.

Far from being the marginal notion to which it is usually reduced, the converse increasing is a non trivial, powerful mathematical tool: even if this goes far beyond Aristotle's intentions, his finite universe equipped with the procedure of converse increasing becomes very close to what in modern term we call a (finite) hyperbolic manifold.

1. Processual infinite

In *Physics* III 4, Aristotle flatly denies the existence of an infinite in act: it cannot be a "something" that exists by itself,[1] like would be Plato's infiniteness, or Pythagoras' *apeiron*, and it cannot be the attribute of a "something" that exist by itself,[2] like Anaximenes' infinite air or Melissus' infinite cosmos.

In the next breath, Aristotle rules out the possibility that the infinite does not exist. After having shown what absurdities both the hypothesis lead to, he says that it is necessary to posit an intermediate party, granting the infinite some form of partial being: the infinite "in a sense is, in another it is not".

Now, the sense in which the infinite "is not" is obvious: the infinite is not in act. Therefore, it must be potentially. But it is necessary to understand what does the phrase "to be potentially" signify when referred to the infinite, since Aristotle explicitly states that one must not take the notion of potentiality here according to its standard meaning of that which is preliminary to actuality.[3]

In fact, Aristotle does not explicitly define this new kind of potentiality: instead he first gives a preliminary idea of the subject

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1 *Ph*. III 4, 202b36 – 203a16; cf .III 5, 204a8 – 34; III 4, 203a6 – 16 and III.1 200b27.

2 *Ph*. III 4, 203a16 – b3; cf III 5, 204b4 – 206a7 and III.1 200b27.

3 When the notions of act and power are referred to a "something", they qualify two different ways of being (a predicate). Now, these ways of being are such that the power to be a certain predicate presupposes and implies being this predicate actually, the act being anterior to the power (*Metaph*. Θ 8, *passim* ; see also *Ph*. II 1, 193b7 – 193b8).

arguing from analogy, and then explains it by means of an example. Given the elusive subject, the analogies are two, mutually exclusive, but complementary: on the one hand, Aristotle assimilates the mode of being of the infinite to that of a day, or a contest; on the other, he assimilates it to that of matter.[4] The example is that of the division of the continuum.

1.1 The day, the contest and the matter

Given their peculiar way of being, one cannot properly say that a day or a contest are in act as totalities. One can say that a day is act during the day, namely in its process of becoming, but in this case what is in act is only a single moment of the day, or a single stage of the contest, or in general a single step of a process. The whole day, or the whole contest or the whole process, can never be said to be when the day, or the contest or the process, are being.

In addition, since a single step in a process is not an object but an action, it is clear that even at the local level one cannot properly maintain that the day and the contest are in act because they are a "something". At most, using Aristotle's words, one can say that they are in act because they are "over and over again something else".

If the way of existence of the infinite is to be graspable from this analogy, it is clear that it must be related to this concatenation of actions: the infinite is not a "something", nor an attribute of a "something", but it could be an attribute of a process. For this reason I prefer to speak of *processual* existence of the infinite, instead of potential.

The critical point, and the reason for which Aristotle introduces a further analogy, is the fact that although the day or the contest cannot be in act as a totality, the notions of a whole day or a whole contest are not in any way contradictory in themselves: since they admit an outcome,

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4 *Ph*. III 6, 206b13 – 16.

that is a last step of their process of becoming, it is sufficient to place oneself beyond it in order to grasp them as totalities.

But it is impossible to place oneself "beyond" the infinite, for – using Aristotle's words – infinite is that "of which there is over and over again something beyond", or even that "of which there is over and over again something to take beyond". More generally – and we come to Aristotle's definition of the infinite – "the infinite is in virtue of taking other and other again, and everything that has been taken is finite, but it is always different".[5]

As immediately seems clear, this is not a conventional Aristotelian definition, in terms of genus and species, or matter and form, but an operative characterization. It does not describe an object but an action – to take something – and the iteration of the action: for each thing we take (*first step*), there is another thing to take (*second step*), and another to take beyond that (*third step*), and another beyond, and so on, over and over again (αἰεί,[6] *next steps*). Speaking in terms of the limit (*peras*) denied (*a* –) in the *apeiron*, for each limit we try to fix, there is something beyond; but this means that there is a new limit to fix, and another "beyond"; and another limit, and another "beyond" and so on, over and over again. It is clear therefore that the infinite cannot be reached: it can be caught only *in* the process which defines it, and from which it is not separable, for

5 *Ph*. III 6, 207a1 – 2; 207a8, and 206a27 – 29, respectively.

6 I understand the adverb αἰεί, usually translated as "always", in the iterative meaning typical of mathematics, where the introduction iterative procedures of demonstration dates back to before Aristotle's time. All these procedures are infinite in the sense defined by Aristotle: they are purely iterative processes, which even in their formal structure are recognizable in Aristotle's definition. See for example the procedure of reciprocal subtraction (*Elements* VII.1 – 2; X.2 – 3) and the so – called *method of exhaustion* (*El*. XII.2, 5, 10, 11 e 12; VIII.9 e IX.34, *Quadratura Parabolae* prop. 20 and its corollary, and prop. 24). On the relation between infinite mathematical procedure of demonstration and Aristotle's definition of potential infinite see Ugaglia 2007. On the meaning of αἰεί in mathematics, see also Mugler, 1958 – 1959, pp. 43 – 44; Federspiel, 2004.

it subsists insofar as this process subsists.[7] In this sense – and we come to Aristotle's second analogy – one can say that the infinite is similar to matter: matter too can be caught only *in* the form that defines it, and from which it is not separable.

In sum, the day and the contest are useful examples because they give an immediate and easily graspable idea of a processual way of being, but they are not the processes we are ultimately interested in. Indeed, they lack the property of going on without limit: both have an end, namely a limit to which one cannot go beyond. To find a good example of a properly infinite process Aristotle turns to something less familiar: the procedure of division of the continuum.

1.2 The infinite division of the continuum

At the very beginning of Book III Aristotle explicitly roots the notion of infinity in that of continuity, which is the place where the infinite primarily reveals itself.[8] Let us see how.

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7 The best characterization of Aristotle's processual infinite I know is the following passage: "all infinity is potential infinity, there is no completed infinite […] the thesis that there is no completed infinity means, simply, that to grasp an infinite structure is to grasp the process which generates it, that to refer to such a structure is to refer to that process, and that to recognise the structure as being infinite is to recognise that the process will not terminate. In the case of a process that does terminate, we may legitimately distinguish between the process itself and its completed output: we may be presented with the structure that is generated, without knowing anything about the process of generation. But, in the case of an infinite structure, no such distinction is permissible: all that we can, at any given time, know of the output of the process of generation is some finite initial segment of the structure being generated. There is no sense in which we can have any conception of this structure as a whole save by knowing the process of generation" (Dummett 1977, pp. 55 – 56). Except that the author is referring to intuitionistic mathematics, not to Aristotle's. I shall develop this point in a forthcoming paper.

8 For a thorough analysis of Aristotle's notion of continuity see white, 1992.

Aristotle defines the continuum as a concept of relation: two terms are continuous when they are in contact and their ends therefore become one.[9] Accordingly, he calls continuous an object composed of continuous parts, that is to say parts that have an end in common.[10] But in order to have an end in common, parts must be congeners,[11] so that they must lose their individuality, and form a homeomeric whole, free of inner limits.[12] But if there are no individual parts, and no actual limits, then there are no parts in act,[13] so that our continuum is potentially divisible everywhere. Hence, using Aristotle's expressions, it is "infinitely divisible"[14] or "divisible into the always (αἰεί) divisible".[15]

It is evident that this last characterization of the continuum – the one Aristotle usually employs – perfectly matches the requirements for being potentially infinite. Take a continuum, for instance a segment: it is divisible (διαιρετόν, *first step*), and every segment one obtains from its division is still divisible (διαιρετόν, *second step*), and every segment one obtains from this second division is still divisible (διαιρετόν, *third step*), and so on, over and over again (αἰεὶ διαιρετά, *next steps*). What we have is not simply the action of dividing, because the absence of any inner hindrance requires us to continue to do so, going beyond any reached division.[16] This is the

9 *Ph.* V 3, 227a11 – 12; see. *Ph.* V 4, 228a29 – 30, *Cat.* 6, 4b20 – 5a14.

10 To be more precise, these parts not only have a limit in common, but are different and locally separated (*Ph.* VI 1, 231b5 – 6, *GC* I 6, 323a3 – 12; *Metaph.* Δ 13 1020a7 – 8).

11 *Ph.* IV 11 220a20 – 21, *Ph.* V 4, 228a31 – b2.

12 *Ph.* 5 IV, 212b4 – 6; *Metaph.* Z 13 1039a3 – 7 see. Δ 26 1023b33 – 34, Z 16, 1040b5 – 8.

13 *GC* I 2, 316a15 – 16.

14 *Ph.* III 1, 200b18; *Ph.* 2 I, 185b10 – 11, VI 6, 237a33, VI 8, 239a22.

15 *Ph.* VI 2, 232b24 – 25; see IV 12, 220a30; VI 1, 231b15 – 16, VI 6, 237b21; VIII 5, 257a33 – 34; *Cael.* 1 I, 268a6 – 7.

16 Aristotle's definition of the infinite contains three requirements: the possibility to take something, over and over again; the condition that what is taken be limited; the condition that it be always different (see section 1.1). As we have

reason why, at the beginning of Book III, Aristotle tied the notions of continuity and infinity.

More generally, the request for infiniteness is satisfied by any procedure that, like the division of the continuum, follows a purely iterative pattern, namely a pattern that is completely defined in terms of an action (or a finite string of actions) and the instruction of repeating it, without limits, going beyond any step reached. Let us think, for example, of a slight variation of the process of division: the process of infinite (division and) subtraction, or decreasing. Given a segment, it is sufficient, after the division, to remove one of the two resulting parts, and this at each step. As in the case of division, the absence of inner limits warrants the possibility of infinitely going on, dividing and subtracting, so that for every given magnitude it is possible to take a smaller one.[17]

This is then the way in which the infinite manifest itself in the continuum: as iteration; namely, as the possibility to go beyond every single act of division. Moreover, this characterization of the infinite suggests the right interpretation of its potentiality: to be potentially, or processually, is the proper form of being of iterative processes, and it is

seen, the first condition is nothing more than the formalization of the way of being "in power" typical of a process. The second condition adds a clarification required in the case of processes that leave residues: take the case of a process that is not reducible to a mere succession of steps but produces, at each step, a "something". The condition that this "something" be always limited rules out the possibility that it might be a case of infinity in act. The third point discriminates between processes that are infinite in a proper sense and processes that are infinite because they are periodic. Since what we obtain by dividing, namely the new segments, is something finite, and different at every step, the procedure of division of the continuum completely fulfills Aristotle's definition.

17 Of course, the process of subtraction is infinite if it is read in this way, as a variant of that division: for in this case, the magnitudes subtracted at each step decrease according to a fixed ratio. On the contrary, if we proceed by subtracting from the segment a constant amount, however small, the process will have an end, as Aristotle correctly observes in *Ph*. 6 III, 206b11 – 12.

exactly the form of being Aristotle was looking for, in order to make his infinite compatible with the finitude of his cosmos.

So, when Aristotle says that a magnitude is potentially infinite, he does not ascribe the attribute "infinite" directly to the magnitude, but to the process of division, which the magnitude, being continuous, supports. This is clear from Aristotle's formulation, when he says that magnitude is infinite by division, or towards the small, if one thinks of the analogous procedure of dividing and subtracting.[18]

But Aristotle also says that, on the contrary, number is infinite towards the more, exceeding in this sense any multiplicity.[19] How is it possible, if in Aristotle's universe things are finite?

1.3 The infinite number

In fact, Aristotle does not relate number primarily to "things" but, once again, to processes. Number does not count the elements of an actual set of things, but the steps of a process. For this reason number is potentially infinite, and the infinity of number is strictly related to that of the division of the continuum.

Aristotle does not spend too many words on this crucial point, except for the passage where, in *Physics* III 7, he briefly explains how the possibility of going on thinking larger and larger numbers depends on the possibility of going on dividing the continuum. The idea is simple: Aristotle constructs numbers as "names" for the divisions of the continuum. Take again our segment, and start to divide it, but now imagine attaching a label to each division; in plain words, imagine counting the steps of the process. Since the process of division is infinite, so is number. But like the process which produce it, number is only potentially (=processually) infinite: since every division implies the possibility of

[18] *Ph*. III 7, 207a32 – b5.
[19] *Ph*. III 7, 207b1 – 3.

a next division, every number implies the possibility of a next number, over and over again, without end.

This conception has an important consequence, for it gives a notion of infinite number that is not separable from the process that defines it. If one stops the procedure of division, of course one obtains a number, namely a numeral, but this number is always a finite one. Once it has been reached, following the process, this finite number can be separated from it, thought of independently and put in relation to something else: the set of the segments produced by division, for example, but also any other set of things – horses, dogs.

To take a concrete example, imagine dividing the segment, and say "one", then divide one of the two segments obtained and say "two", then… and say "seven". Seven is the number of the divisions made, but also of the segments obtained (minus one), of the seven pencils I have on my desk, of the Pleiades.

Of course, by going on with the process one can produce, and separate from it, larger and larger finite numbers, but the action of separating them, making them actual numerals, is subordinate to the action of reaching their corresponding step of the division, and since the procedure is infinite, there is no a final step to reach, nor any final number to associate to it. So, like every other infinite in Aristotle's system, number too is infinite only in power, not in act.

The crucial point to observe is that this idea of potential (=processual) infinite number allows the physicist to carry out any sort of operation without contravening Aristotle's firm denial of the actual infinite. The fact that numbers are infinite is not made to depend or imply the actual existence of infinite things to be numbered (the segment *is not* made up of points that can be counted), but the continuum being infinitely divisible, there is a potentially infinite series of actions to be counted, and this is enough.

1.4 The essential properties of the infinite: locality and timelessness

In a purely iterative process the possibility of infinite repetition – that is, iteration in the proper sense – ultimately depends on the equivalence between the different steps constituting the process: each step being indistinguishable from the preceding step and from the following one, the process can go on indefinitely:[20] divide, then divide, then divide... or: divide and subtract, then divide and subtract, then divide and subtract…

This has two main consequences: Aristotle's notion of infinite is a purely local concept, and it does not depend on time.

Concerning the first point, I am simply using the adjective "local" to express the fact that, given the equivalence just mentioned, in a purely iterative procedure everything hinges on the concatenation between two actions, so that there is no need for notions such as that of *result* of the process or of the process in its *entirety*. The "whole" process can be summed up as a prescription to perform an action, or a finite string of actions, together with the instruction of repeating it: in the case of the division of the continuum, this means a law, the so – called *law of convergent series*. Indeed, it is not necessary, and of course impossible, to describe the whole procedure: divide at the middle (or at one third, or wherever you want), now divide at the middle, now divide at the middle... but it is sufficient to say: "at whatever step you have arrived, divide at the middle what you have in front". This is what I mean by saying that the infinite is a local property.

In addition, an infinite so defined, while being dependent on a process, is nevertheless completely independent of time: it is local because everything hinges on the concatenation between two specific actions, but it is timeless because this concatenation amounts to a logical rather than chronological dependence. Two steps in an iterative process will occur one *before* and the other *after* in a logical sense, and this logical

20 Note that in the case of the continuum the ultimate cause of the possibility of indefinite iteration is a physical cause, namely the absence of inner limits. For Aristotle it is not sufficient to possess the idea of succession (see the next note), in order to obtain that of indefinite iteration: he needs both the notion of succession and the material absence of physical obstructions.

succession is only accidentally realized in time: when an iterative process involves physical objects, it is obviously actualized through physical actions, which necessarily take place over time, but the fact that the process does not come to an end is not a question of time.[21]

We are thus in possession of everything we need in order to affirm that such a local and timeless infinite, tied to the notion of process yet totally free from any idea of an actual result, is entirely compatible with Aristotle's cosmos: its processual existence – or its being "potentially", to quote Aristotle himself – receives (almost) no restriction from the spatial limitation imposed to the cosmos by the size of the external sphere.

2. Finite physics and finite mathematics

Until now we have discussed in detail only one form of potential infinite: the procedure of division of the continuum. This is reasonable since all the manifestations of the infinite allowed in Aristotle's cosmos are to some degree traceable to the process of division of the continuum: some of them are mere variations on the theme of the continuum, like number, while others, not directly based on it, are nonetheless analysable in terms of iterative procedures.

In particular, one can analyse in term of iteration the motion of the heavens, which is nothing but a periodical motion – namely a particular

[21] Aristotle gets to the notion of mathematical a – temporal succession – which here I have called "logic" – starting from the notion of before/after in place, namely from the teleological order of the world, deprived from any physical feature (see his discussion of time in *Ph*. IV 10 – 14). In contrast, the idea that for Aristotle the process of division does depend on time is upheld by Wieland 1970 and Hintikka 1973 (see, however, the criticism advanced in Lear 1979), who follows a strictly temporal reading of the adverb ἀεί. Again, an interesting parallel can be made between the teleological foundation of Aristotle's notion of succession, which grounds his idea of the infinite, and Brouwer's temporal foundation of the same notion (see n.7).

form of infinite iterative process – [22] and once one has accepted the infinite belonging to the motion of the heavens, one must accept all the other manifestation of the infinite directly traceable to it: namely time, which is the number of motion,[23] and the process of coming to be and passing away of men, animals and the sub – lunar world in general, which for Aristotle imitates the movement of the heavens.[24]

All these forms of infinite are sufficiently "finite" to be compatible with the finite size of the universe, and sufficiently "infinite" to permit the development of a consistent physical theory, as Aristotle's one. Continuity of magnitude, movement and time, which lies at the basis of Aristotle's system, involves a perfectly acceptable notion of infinite, and so do the notion of number and that of the eternity of motion, time and generation.

But there are other manifestations of the infinite, seemingly innocuous, which are notwithstanding forbidden, and must then be excluded from physics: so are all these processes that, despite being in themselves only potentially infinite, do in fact involve an actually infinite "something". Or even just a finite "something", bigger than the universe. The main example of forbidden infinite is the process of increasing the magnitude.

2.1 No infinitely extendible physical magnitudes

Given the local character of Aristotle's potential infinite, one might conclude that it would be the same to go on dividing a continuum

[22] *Ph* IV 4, 211a13 – 14; VIII 10, 267b11 – 17. Strictly speaking, this kind of infinite does not completely fulfill Aristotle's definition (see note 16) because after a certain number of steps one comes back to something already taken. In this sense, infinite circular motion is something more perfect than the proper "linear" infinite. Like the proper infinite, it always has something beyond it, but unlike the proper infinite it does not absolutely lack this "something": in some sense, the latter is already present within it.

[23] *Ph* VIII 1 251b10 – 28.

[24] *GC* II 10, 336a15 – 18; see also 336b1 – 10, *Ph*. II 1, 193a27 – 28, *Cael.* II 3, 286b5 – 9, and *Gen. An*. II 1, 731b32 – 732a1. On this argument see Quarantotto 2005.

in an infinite cosmos or in a finite one. Since the infiniteness lies in the process, and not in its result, one might in fact reasonably suppose that the effective limit of the cosmos does not interfere with it. An analogous conclusion seems to be applicable to any infinite procedure deduced from the division of the continuum. In particular, it holds for the process of (dividing and) subtracting the segment, shortening it over and over again, so that it seems reasonable to think that it holds also for the procedure of (multiplying and) adding the segment, increasing it over and over again. On the contrary, Aristotle tells us that it is not so: given a finite magnitude, it is not always possible to take a larger one, because at a certain point, unavoidably, one runs into a physical impossibility.

Imagine taking a segment and doubling it. Now double this result, then double the new result and so on: however small the starting segment may be, in a finite number of steps the size of the cosmos is reached, and here the process will be forced to stop. Not because there is no place to go beyond the cosmos, but because there is no "beyond" the cosmos: once the size of the cosmos has been reached, the action itself of going beyond loses all meaning. Therefore, the step that reaches the size of the universe must necessarily be the last: the iteration is broken; the last step means an outcome of the process, and this means that the procedure is not infinite.

The situation is radically different from the one occurring in the case of decreasing. In this case, even if no final result – namely an actually infinitely small magnitude – can ever be reached, such an infinitely small magnitude is not a physical impossibility: it does not exist only *because* it would be the "result" of a process that by definition has no result. But the process itself exists, so that Aristotle can say that towards the small there is no infinite in act (=the result of the infinite process of shortening), but there is in power (=the infinite process of shortening itself). In more concrete terms, in Aristotle's cosmos there are no infinitely small magnitudes, but any magnitude is infinitely reducible.

In the case of increasing, on the contrary, the hypothetical final result of an infinite procedure – namely an actually infinitely large magnitude – cannot be reached for physical reasons: not only because it would be the "result" of a process that by definition has no result, as was the case for the infinitely small magnitude, but because no magnitudes exist – be they finite or infinite – bigger than the fixed size of the universe. As Aristotle repeatedly states, towards the greater there is no infinite: neither in act (=the result of the infinite process of increasing), nor in power (=the infinite process of increasing itself). In more concrete terms, in Aristotle's cosmos there are no infinitely extended magnitudes, nor infinitely extendible ones, and this result has fatal consequences for Aristotle's mathematics.

2.2 No infinitely extendible mathematical curves

Why would the conclusion now reached, that in Aristotle's physics there are neither actually infinite magnitudes nor finite magnitudes that are infinitely extendible, be a problem for a mathematician? Even though they are not in the sublunary world, what could prevent the mathematician from imagining actually infinite lines? or at least from infinitely extending the finite ones?

Indeed, nothing prevents us from doing so: when in contemporary mathematics we speak about straight lines, without any specification, it is usually understood that we mean infinitely extended straight lines; we construct segments as finite portions of straight lines, and we take the possibility of the infinite extension for granted, without questioning it.

But nothing prevents the Greek mathematicians – although they usually take as their primary notion that of a finite straight line – from accepting the possibility of extending it without limits, as the following statements from Euclid's *Elements* make clear:

> To produce a finite straight line continuously in a straight line. [25]
> Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either directions.[26]

In particular, nothing prevents Aristotle's master Plato, who conceives his objects of study as separate and independent from the physical world. And nothing would prevent Aristotle either, if he were no more a mathematical "immanentist" than he is a physical "finitist".

Indeed, the problem arises when, as in the case of Aristotle, the mathematician is not required to deal with separate mathematical objects but with natural ones: the same objects the physicist deals with, albeit from a different perspective. As Aristotle stresses on several occasions, in his system there is a perfect coincidence between the objects of study of physicists and those of mathematicians,[27] with the only difference that the former study them as changing objects, while the latter do not consider them as changeable, but disregard change.[28]

Therefore, although the mathematician does not consider physical objects in their entirety but acts upon them, in order to put aside their more strictly physical features,[29] he is conditioned by certain basic properties, which cannot be erased by a simple change of perspective.[30]

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25 Euclid, *Elements* Post. I.3 (Heath's translation).

26 Euclid, *Elements* Def. I.23 (Heath's translation).

27 *Metaph* M 3, 1077b22 – 1078a9; *Ph*. II 2, 193b23 – 194a12; cf. *de An*. I 1, 403a15 – 16; *Metaph*. N 2 1090a13 – 15.

28 *Metaph*. E 1, 1026a14 – 15 *et seq*.

29 Aristotle is rather vague about the effective ways of establishing such a crucial preliminary step, called ἀφαίρεσις (abstraction, literally "removal": see Philippe, 1948; Cleary, 1985; Mueller, 1990).

30 Aristotle says that a bronze ball touches a physical straight line at a single point (*de An*. 1 I, 403a12 – 15; about the apparently contradictory passage in *Metaph*. B 2, 997b34 – 998a6, the aporetic context suggests a reference to the opinions of Aristotle's opponents, and not to his own point of view). For a similar interpretation of Aristotle's philosophy of mathematics, see Lear, 1982; among the

The problem is not that the mathematician is unable to imagine a magnitude greater than that of the cosmos, since Aristotle himself makes use of infinite straight lines, when he hypothetically assumes the infiniteness of the universe, in order to disprove it.[31] Rather, the point is that it is not through such objects of thought that one can practice mathematics, just as it is not with *hircocervi* or sphinxes that one practices physics. The mathematician must deal with objects that not only can be thought, but can be thought *as* physical objects, namely objects whose conceivability is not in conflict with the physical structure of the cosmos. But as we have just proved, in Aristotle's cosmos there are no infinite straight lines: neither actually infinite straight lines nor infinitely extendible straight lines.[32]

So how can Aristotle nonetheless maintain that the mathematician is in no way affected by this requirement?

3. Finite but hyperbolic?

The answer is simple: to deal with a finite magnitude as though it were infinite. Of course, one has to possess the necessary equipment for doing so, and Aristotle has it. To put it more accurately, he constructs

many supporters of the opposite view, see for example Mueller, 1970. More generally on mathematics in Aristotle see Cleary, 1995 and Mendell, 2008.

31 See for example *Cael*. I 5, 271b28 – 272a7, where Aristotle adopts the hypothesis of an infinite sky, and infinitely extends a line within such a model (cfr. *Top*. 148b30 – 32).

32 The false claim that a straight line can be indefinitely extended, thus obtaining a mathematical object in the Aristotelian sense, is expressed in a very clear form already in Simplicius and Philoponus (*in Ph*., 512.19 – 36 Diels; *in Ph*., 482, 28 – 483,14 Vitelli). It seems that a completely different opinion must be ascribed to Alexander, but unfortunately the very few extant passages (*in Ph*., 511, 30 – 512.9 Diels) do not allow us to settle the question. See however Rashed 2011 on Alexander's mathematical ontology, which indeed seems to require such an interpretation.

such an equipment in *Physics* III 6 – 7, in the form of an iterative procedure, which he calls converse increasing. In the next section I will briefly describe the procedure, then I will show how it works, considering the case of the asymptotic properties of a curve.

3.1 The converse increasing

In section 2.1 we have discussed Aristotle's denial of the procedure of infinite increase of magnitudes. To be more precise, what Aristotle denies is a procedure that tends, as its hypothetical result, towards an infinite magnitude: in this case not only the actuality, but also the potentiality of the infinite must be denied. But not all the procedures of increasing are of this kind: in fact, there is a very crucial exception that wraps – up the whole story. Aristotle calls it "converse increasing", because – as we will see in a moment – he obtains it by conversion,[33] starting from a process of infinite subtraction:

> To exceed every ‹magnitude› by addition is not possible even potentially unless there is something which is actually infinite, accidentally, as the natural philosophers say that the body outside the cosmos, of which the substance is air or some other such thing, is infinite. But if it is not possible for there to be a perceptible body which is actually infinite in this way, it is manifest that there cannot be one even potentially ‹infinite› by addition, except in the way that has been stated, conversely to the division – process (*Ph* III 6, 206b20 – 27).

Indeed, some line before Aristotle has introduced a procedure of increasing which, while being infinite, tends towards a finite magnitude. The idea is very simple: take a finite magnitude, a segment for example, divide it into two parts and keep one of them. After that, divide the other

[33] The Greek term is ἀντεστραμμένως, from ἀντιστρέφω (cf. 206b27, 207a23), in *Prior Analytics* the verb is used in a technical sense for the procedure of conversion of a premise.

into two parts, take one and add it to the one set aside in the previous step. After that, divide the other into two parts, take one and add it to the one set aside at the previous step, and so on, over and over again. To each step in the procedure of division there corresponds a step in the converse procedure of addition, and since the process of dividing the magnitude is infinite, the converse process of increasing too must be infinite. This notwithstanding, this infinite process does not lead "beyond" any magnitude, for it tends towards "a definite amount", namely the initial segment:

> The ‹infinite› by addition is in a sense the same as that by division. For in that which is finite it comes to be by addition, conversely: just as something is seen as being divided ad infinitum, in the same way it appears to be added to a definite amount. For if, in a finite magnitude, one takes a definite amount and takes in addition in the same proportion,[34] not taking a magnitude which is the same with respect to the whole, one will not traverse the finite magnitude (*Ph*. III 6, 206b3 – 9).

It is clear that this procedure of increasing is completely different from the one described in section 2.1 and consisting in adding constant quantities: for in this case, however great the finite magnitude one fixes as a final limit, and however small the constant finite quantity one adds at every step, the former quantity will always be exhausted within a finite number of steps, as Aristotle correctly observes:

> but if one increases the proportion so that one always takes in the same particular magnitude, one will traverse it, because every finite quantity is exhausted by any definite quantity whatever (206b9 – 12).

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34 Aristotle's request that the ratio between the whole and the part that is subtracted at each step be a constant is not a necessary condition: whatever the ratio, the process will still be infinite. The addition of this condition makes it quite clear that Aristotle is thinking of an iterative process, each step of which is perfectly identical to the previous one and can therefore be completely defined once and for all (for example by the rule: "subtract each time half").

3.2 The infinite at the finite

Having the procedure of converse increasing in mind, it is easy to understand the very cryptic passage, which concludes Aristotle's analysis of the infinite, and where Aristotle explains why the refusal of any infinite extension is not a problem for the mathematicians. It is not a problem, Aristotle says, because the mathematicians do not use actually infinite magnitudes, but only increases of magnitude: finite increases "as great as they want":

> This reasoning does not deprive the mathematicians of their study, either, in refuting the existence in actual operation of an untraversable infinite in the direction of increase. Indeed, they do not need the infinite, for they make no use of it; they only need there to be a finite ‹increase› as great as they want (207b27 – 31).

The requirement that the increase be finite excludes that it may be the (unattainable) result of a process of infinite increasing, which Aristotle has explicitly excluded. Instead, it will be – and must be regarded as – the (equally unattainable, but finite) result of an infinite process of converse increasing. Indeed, as we have seen, in this case it is the procedure – namely the converse increasing – that is infinite, not the global increase.[35] In other words, to have segments plus a procedure of converse

[35] I take the word "increase" at line 29 as the implied subject of "as great as they want" at line 31. Traditionally, this claim has been referred to the straight line, so that the passage is read as an explicit reference to *El.* VI.10. I prefer to avoid referring to any straight line because given the absence of any mention of lines in the whole preceding argument, this reading seems to me to be a consequence of – rather than evidence for – an interpretation in terms of *El.* VI.10. Alternatively, one can maintain the reference to the straight line: in this case however it must be clear that this finite straight line must be considered not as a *per se* object but as the (unattainable) result of a process of inverse increasing. With a straight line as long as he wishes it to be, but without the idea of an infinite process of converse increasing, the Aristotelian mathematician does not solve the problem of the infinite. Indeed, he falls into a process of unconditioned infinite increasing, of which the finite magnitude is only a step.

increasing, as it is the case for Aristotle, is mathematically equivalent to have infinitely extended lines: what one can do in the second case, one can do in the first too.

Regarding common shortcomings, in either case one cannot make use of an actual infinite, because both the converse increasing and the generic extension are infinite only in a potential sense. But this is not a problem: after all, no mathematician made use of the actual infinite before Cantor, or at least before the 17th century.

Concerning the positive properties, the best way to ascertain the equivalence of the two models is to contrast them in relation to asymptotic properties, namely the behaviour of the curves "at the infinite" (parallel lines, the asymptotes of a hyperbola...). Let us consider, for example, proposition II.14 of Apollonius' *Conics*, where we read that "the asymptote and the section, if infinitely extended, come closer to one another and leave an interval which is smaller than any given interval".

Apollonius' proof consists in fixing an interval *K*, and constructing a suitable strictly decreasing succession, namely the succession of the segments cut off on a sheaf of parallel lines incident on the asymptote, by the intersection with the hyperbola and the asymptote. Since the succession of the segments is strictly decreasing, and the continuum is infinitely divisible, at a given point the segments cut off will eventually become smaller than *K*. The smaller is *K*, the farther from the origin the intersection between the parallel line containing the segment and the asymptote is located, so that this intersection tends to the infinite for *K* tending to zero.

Now imagine replacing the point at the infinite with a point on the boundary of Aristotle's cosmos, and the procedure of infinite increasing which tends to it with a procedure of converse increasing which tends to the limit of the cosmos: it is easy to see that what in the first case was imagined to happen "at the infinite" is now happening at the edge of the cosmos. In this case, for *K* tending to zero, the intersection tends to the boundary of the cosmos.

In fact, the only important thing for the purpose of the demonstratio is to have at our disposal an infinite number of steps. It matters little whether these steps are related to a process of infinite increasing that leads away indefinitely, or to a process of converse increasing, which in infinitely many steps leads to the finite. [36]

But – one might have reason to wonder – what about magnitudes greater than the size of the cosmos? Suppose a mathematician posits a right line longer than the size of the cosmos: he is allowed to do so by Aristotle himself, who speaks about a finite increase as great as the mathematician wants. Although it is finite, this line is not among the existing magnitudes: what can the Aristotelian mathematician do with it?

Once again the answer lies in the process of converse increasing. Being finite, the magnitude can always be rescaled (= reduced in proportion) to another magnitude,[37] lesser or equal to the size of the cosmos, which is the maximal existing magnitude. The mathematician can now work with this new magnitude, which can be thought of as the (unattainable) result of a new process of converse increasing, every step of which is rescaled with the same proportion.

Take for example a segment AB, and suppose it is greater than the size of the cosmos. But also suppose that by reducing it three times one obtains a segment CD, of the dimension of the cosmos: the process of converse increasing described at the beginning of this section must apply to the segment CD exactly as it does to AB, with the only difference that every added segment will be three times smaller than the one added at the same step of the original process. By reading CD as its (unattainable) result, the mathematician can always work with "real" magnitudes, that is with magnitudes contained within the cosmos:

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36 The characterization of the asymptotes of the hyperbola too, which are expressed in terms of the shape parameters of the hyperbola, is a purely local characterization. It involves the infinite in a form perfectly compatible with the Aristotelian constraints (see *Con*. II.1).

37 Of course, in order to speak of rescaling, both magnitudes must be finite (see the next note).

> Another magnitude of any size whatever can be divided in the same proportion as the maximal magnitude;[38] so that, at least for the purpose of the proof, it will make no difference, and concerning its being it will be among those magnitudes that exist (207b31 – 34).[39]

The size of the cosmos is indeed a maximum for existing magnitudes. It is not a maximum for only conceivable magnitudes, and this legitimizes the transmitted form of the last sentence: any other magnitude can be cut in the same ratio as the maximal one, no matter whether it (i.e. the other

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[38] It is important to note that this sentence *does absolutely not* mean – as is often mistakenly maintained – that it is sufficient to rescale an infinitely extended line, in order to make it finite. While a property that holds in a point to the finite can be transported to a nearer point by appropriate rescaling – a shrinking of the design, roughly speaking – there is *no* change of scale that can bring an asymptotic property closer, that is to the finite. No matter how small the drawing, the point "at the infinite" will always lie outside, by definition. On the possibility of rescaling the infinite see for example Hussey, 1983, p. 95: "Thus, instead of saying that a hyperbola approaches a straight line as it tends to infinity, one may say that, taking sufficiently small scale models of the original hyperbola, we shall find the scale models approaching arbitrarily closely to the corresponding lines". See too the first occurrence of the claim in Hintikka, 1973, p. 119. Baseless criticisms of Aristotle's system grounded on this kind of "underestimation" of the mathematical skills of the philosopher are very common: see for example Knorr, 1982 and Milhaud, 1903.

[39] Following the standard interpretation of this passage, which ignores converse increasing, the maximal magnitude is a magnitude as large as the mathematician wants it to be, and "every other magnitude can be cut in the same ratio as this maximal magnitude, and at least for the purpose of proof it will make no difference if it is between the existing magnitudes". The rescaled one will be. Unfortunately, a line as large as one wants cannot be a maximum: by definition the maximum element of a set is an element such that any other element of the set is lesser or equal to it. If a magnitude can be as large as the mathematician wants, for every given magnitude he can take a larger one, so that no magnitude can be the maximal one. Moreover, to allow this interpretation, the transmitted text – on which all the manuscripts agree, with minor variations – must be modified by expunging two words, as in Ross 1936 (οὐδὲν διοίσει τὸ [δ'] εἶναι ἐν τοῖς οὖσιν [ἔσται] μεγέθεσιν).

magnitude, the conceived one) ranks among existing magnitudes – the rescaled one certainly will.

4. Concluding remarks

Aristotle's finite cosmos, equipped with the infinite process of converse increasing, may seem like a hyperbolic manifold to contemporary mathematicians: a bounded space – a sphere, for example – equipped with a metric that makes the boundary unattainable: as the boundary is progressively approached, the lengths shorten, so that the path towards the limit becomes infinite.

This is exactly the idea which underlies Aristotle's introduction of converse increasing: dealing with a finite magnitude as though it were infinite. Imagine trying to reach the boundary of a magnitude by means of a process of converse increasing: since the process of going through it is infinite, the limit is unattainable, as in the hyperbolic manifold.

Of course, Aristotle does not introduce the idea of infinite converse increasing with the intention of establishing a new mathematics, but simply in order to exit the *impasse* he got into in an attempt to reconcile finitism and immanentism.

Actually, the trick of converse increasing enables Aristotle to avoid the *impasse* at a local level, but at the same time it raises a serious problem at a global one. If properly developed, the idea leads to the described model of non – Euclidean hyperbolic geometry, which would force Aristotle to abandon nearly all the geometric results that he needs (results obtained, of course, in the context of Euclidean geometry). Now, it is clear – and perfectly understandable – that Aristotle is far from perceiving the problem and therefore continues to believe that the sum of the interior angles of a triangle is equal to two right angles, and that this may be used as an example of a necessary truth.

In general, the idea that mathematical objects coincide with physical ones, and that there is only one cosmos (which, by definition, contains the totality of existing things), implies that there is only one mathematics, and hence only one geometry. Even when Aristotle states that in mathematics one deals with necessity *ex hypothesis*, namely as a kind of necessity depending on the first principles of geometry, he is not anticipating the possibility of other geometries, depending on other sets of principles, but is simply pointing out what the direction of necessity is: given certain principles, the results are obtained by necessity. Even if other sets of principles can be conceived, they have nothing to do with geometry – which is *the* unique geometry of the unique cosmos.

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