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A NOTE ON THE GENERALIZED BI-PERIODIC LUCAS-BALANCING NUMBERS

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Abstract. In this study, we introduce a new class of integers called the sequence of generalized bi-periodic Lucas-balancing numbers, which extends the well-known sequence of Lucas-balancing numbers. We present several fundamental properties, including the deduction of the corresponding generating function, as well as homogeneous and non-homogeneous recurrence relations associated with this new sequence. We also formulate generalized versions of Binet's formulas for these numbers. In addition, we investigated the validity of several classical identities within this new context, such as the Tagiuri-Vajda, d'Ocagne, Catalan, and Cassini identities, considering the two different cases of the discriminant value of the equation polynomial associated with the recurrence relation. These extensions contribute to a structural and algebraic deepening of the properties of Lucas-balancing numbers.

Keywords . Lucas-balancing sequence, bi-periodic sequence, identities, analytic representations.

UMA NOTA SOBRE OS NÚMEROS BI-PERIODICOS LUCAS-BALANCEADOS GENERALIZADOS

Resumo. Neste estudo, introduzimos uma nova classe de números inteiros denominada sequência dos números biperiódicos Lucas-balanceados generalizados, que amplia a conhecida sequência dos números Lucas-balanceados. Apresentamos diversas propriedades fundamentais, incluindo a dedução da função geradora correspondente, além de relações de recorrência homogêneas e não homogêneas associadas à nova sequência. Também formulamos versões generalizadas das fórmulas de Binet para esses números. Além disso, investigamos a validade de várias identidades clássicas dentro desse novo contexto, como as identidades de Tagiuri-Vajda, d?Ocagne, Catalan e Cassini, considerando dois casos distintos dependentes do valor do discriminante da equação característica associada a recorrência. Essas extensões contribuem para o aprofundamento estrutural e algébrico das propriedades dos números Lucas-balanceados.

Palavras-chave. Sequência de Lucas-balanceados, sequência bi-periódica, identidades, representação analítica.

UNA NOTA SOBRE LOS NÚMEROS BI-PERIODICOS LUCAS-BALANCEADOS GENERALIZADOS

Resumen. En este estudio proponemos una nueva clase de números enteros denominada sucesión de números biperiódicos generalizados Lucas-balanceados, que amplía la conocida sucesión de números Lucas-balanceados. Presentamos varias propiedades fundamentales, incluida la deducción de la función generatriz correspondiente, así como relaciones de recurrencia homogéneas y no homogéneas asociadas a la nueva sucesión. También formulamos versiones generalizadas de las fórmulas de Binet para estos números. Además, investigamos la validez de varias identidades clásicas dentro de este nuevo contexto, como las identidades Tagiuri-Vajda, d'Ocagne, Catalana y Cassini, considerando dos casos diferentes. Estas extensiones contribuyen a una profundización estructural y algebraica de las propiedades de los números Lucas-balanceados.

Palabras clave. Sucesión del Lucas-balanceados, sucesión bi-periódica, identidades, representación analítica.

1 Introduction

Integer sequences are an interesting subject of research because of their applications in various areas of science. This topic is explored from several perspectives, such as matrix, combinatorial, and analytical perspectives. Consequently, a wide range of sequences has been introduced and generalized. One of these generalizations is the bi-periodic sequence. The bi-periodic sequences, such as the well-known bi-periodic Fibonacci and bi-periodic Lucas sequences, extend the concept of traditional mathematical sequences in terms of index parity.

The first case that appears in the literature is the bi-periodic Fibonacci sequence, introduced by Edson and Yayenie in [1]. The authors denoted the sequence as $\{F_n^{(a,b)}\}_{n\geq 0}$ and defined by

$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & if \ n \ is \ even, \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & if \ n \ is \ odd, \end{cases}$$

for any real nonzero numbers a and b and $n \geq 2$, with initial conditions given by $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. Therefore, this definition and these results were extended to other sequences:

1. Bilgici introduced the bi-periodic Lucas sequence in [2],

2. Uygun and Karatas introduced the bi-periodic Pell–Lucas numbers in [3],

3. Uygun and Owusu introduced the bi-periodic Jacobsthal numbers in [4], and,

4. Catarino and Spreafico introduced the bi-periodic Leonardo sequence in [5].

In recent years, a number of studies have emerged focusing on a wide variety of sequences. In these studies, the introduced bi-periodic sequence was analyzed, and its recurrence relations, properties, generating functions, and Binet's formula were established.

In this article, we aim to generalize both the sequence of Balancing numbers and the sequence of Lucas-balancing numbers. The concept of balancing numbers was first introduced by Behera and Panda [6] in 1999 in connection with a Diophantine equation. It consists of finding a natural number n such that

$$1 + 2 + 3 \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some natural number r. In this context, n is called the balancing number and r is the balancer. Denote $\{B_n\}_{n\geq 0}$ the sequence of balancing numbers. In [6], Behera and Panda first obtained the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1},$$

for $n \geq 1$, and with initial terms $B_0 = 0$ and $B_1 = 1$, and then developed the Binet formula, generating function and several identities by solving this recurrence relation as a second-order linear homogeneous difference equation. It is well known that b is a balancing number if and only if the number $8b^2 + 1$ is a perfect square, according to Ray [7] this motivated the definition of the Lucas-balancing numbers. Denote the sequence $\{C_n\}_{n\geq 0}$ of the Lucas-balancing numbers corresponding to the balancing numbers $\{B_n\}_{n\geq 0}$ defined by

$$C_n = \sqrt{8(B_n)^2 + 1}.$$

It is interesting to note that the sequence of Lucas-balancing numbers satisfies the same recurrence relation as that of balancing numbers, namely,

$$C_{n+1} = 6C_n - C_{n-1},$$

for $n \geq 1$, and initial conditions $C_0 = 1$ and $C_1 = 3$ (see more in [8, 9, 10, 11, 12, 13, 7, 14]).

Tasci and Sevgi [15] introduced the bi-periodic Balancing numbers $\{B_n^{(a,b)}\}_{n\geq 0}$, defined recursively by, for any two non-zero real numbers a and b,

$$B_0^{(a,b)} = 0,\ B_1^{(a,b)} = 1,\ B_n^{(a,b)} = \begin{cases} 6aB_{n-1}^{(a,b)} - B_{n-2}^{(a,b)}, & \text{if } n \text{ is even} \\ 6bB_{n-1}^{(a,b)} - B_{n-2}^{(a,b)}, & \text{if } n \text{ is odd} \end{cases},\ n \geq 2\,. \tag{1}$$

Motivated by prior research and the notion of bi-periodic balancing numbers, we propose to introduce and explore a new sequence, namely, the generalized bi-periodic Lucas-balancing numbers, with arbitrary initial conditions.

This article is organized as follows: the next section defines the bi-periodic balancing numbers and the generalized bi-periodic Lucas-balancing numbers and discusses their properties. Section 3 introduces the generating functions associated with these sequences. Section 4 provides the Binet formulas for the two distinct cases. Section 5 is focused on deriving various classical identities related to these sequences: Tagiuri-Vajda?s, d?Ocagne?s, Catalan?s, and Cassini?s identities. Finally, some conclusions are stated.

2 The generalized bi-periodic Lucas-balancing numbers

In this section, we will introduce the bi-periodic Lucas-balancing numbers, and the generalized bi-periodic Lucas-balancing numbers. Consider the following definition.

Definition 1. For any non-zero real numbers a, and b, the sequence of bi-periodic Lucas-balancing $\{C_n^{(a,b)}\}_{n\geq 0}$ is defined recursively by

$$C_0^{(a,b)} = 1,\ C_1^{(a,b)} = 3,\ C_n^{(a,b)} = \begin{cases} 6aC_{n-1}^{(a,b)} - C_{n-2}^{(a,b)}, & \text{if } n \text{ is even} \\ 6bC_{n-1}^{(a,b)} - C_{n-2}^{(a,b)}, & \text{if } n \text{ is odd} \end{cases},\ n \geq 2\,. \tag{2}$$

The first six elements of the bi-periodic Lucas-balancing numbers are present in Table 1.

Table 1: bi-periodic Lucas-balancing

$C_0^{(a,b)}$	1
$C_1^{(a,b)}$	3
$C_2^{(a,b)}$	$18a-1$
$C_3^{(a,b)}$	$108ab-6b-3$
$C_4^{(a,b)}$	$648a^2b-36ab-36a+1$
$C_5^{(a,b)}$	$3888a^2b^2-216ab^2-324ab+12b+3$

when $a = b = 1$, we have the classic Lucas-balancing numbers. If we set $a = b = k$, for any positive number, we get the k-Lucas-balancing numbers.

Since Lucas-balancing numbers are given by the same recurrence relation of the Balancing numbers but with different initial conditions, then these sequences can be extended to any initial conditions, as defined below.

Definition 2. For any four real numbers a, b, c and d, with a and b non-zero, the sequence of generalized bi-periodic Lucas-balancing numbers $\{z_n\}_{n\geq 0}$ is defined recursively by

$$z_0 = c,\ z_1 = d,\ z_n = \begin{cases} 6az_{n-1} - z_{n-2}, & \text{if } n \text{ is even} \\ 6bz_{n-1} - z_{n-2}, & \text{if } n \text{ is odd} \end{cases},\ n \geq 2\ .$$

The first five elements of the generalized bi-periodic Lucas-balancing numbers are exhibited in Table 2.

Table 2: generalized bi-periodic Lucas-balancing

z_0	z_1	z_2	z_3	z_4
c	d	$6ad - c$	$36abd - 6bc - d$	$216a^2bd - 36abc - 12ad + c$

Observe that when $c = 1$, and $d = 3$ we have that $\{z_n\}_{n\geq 0} = \{C_n^{(a,b)}\}_{n\geq 0}$. If we set $c = 0$, and $d = 1$, we have that $\{z_n\}_{n\geq 0} = \{B_n^{(a,b)}\}_{n\geq 0}$.

The next result follows directly from the Definition 2.

Proposition 1. The generalized bi-periodic Lucas-balancing sequence $\{z_n\}_{n\geq 0}$ satisfies the following properties:

(a) $z_{2n} = (36ab - 2)z_{2n-2} - z_{2n-4}$;

(b) $z_{2n+1} = (36ab - 2)z_{2n-1} - z_{2n-3}$.

Proof. Using the recurrence relation for the generalized bi-periodic Lucas-balancing numbers we can obtain:

$$\begin{aligned} z_{2n} &= 6az_{2n-1} - z_{2n-2} \\ &= 6a(6bz_{2n-2} - z_{2n-3}) - z_{2n-2} \\ &= 36abz_{2n-2} - 6az_{2n-3} - z_{2n-2} \\ &= 36abz_{2n-2} - (6az_{2n-3} - z_{2n-4}) - z_{2n-2} - z_{2n-4} \\ &= (36ab - 2)z_{2n-2} - z_{2n-4}\ . \end{aligned}$$

This proves the item (a). The proof of item (b) is done similarly. □

As a consequence of the Proposition 1, when we subtract z_{2n} from z_{2n+1}, we obtain a homogeneous recurrence relation of 5th order.

Theorem 1. *The generalized bi-periodic Lucas-balancing sequence* $\{z_m\}_{m\geq 0}$ *satisfies the homogeneous recurrence relation*

$$z_{m+1} = z_m + (36ab-2)z_{m-1} - (36ab-2)z_{m-2} - z_{m-3} + z_{m-4},\ m \geq 4.$$

Proof. One way is in Proposition 1, let us make a difference between (b) and (a), and changing $2n$ by m. In a similar way, let us consider the difference between (a) and (b), and change n by $m+1$. □

Now, by considering the parity function $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$, that is, $\xi(n) = 0$ when n is even, and $\xi(n) = 1$ when n is odd, we can rewrite Equation (2) and establish another homogeneous recurrence relation.

Lemma 1. *The generalized bi-periodic Lucas-balancing numbers* $\{z_n\}_{n\geq 0}$ *satisfy the following properties:*

$$z_{n+2} - 6a^{1-\xi(n)}b^{\xi(n)}z_{n+1} + z_n = 0\ . \tag{3}$$

3 Generating function

This section will provide the generating function for the generalized bi-periodic Lucas-balancing numbers. First, consider the following auxiliary result.

Lemma 2. *Let* $\{z_{2j-1}\}_{j\geq 1}$ *be the generalized bi-periodic Lucas-balancing numbers so the generating function is* $\sum_{j=1}^{\infty} z_{2j-1}x^{2j-1} = \dfrac{dx + (d-6bc)x^3}{1-(36ab-2)x^2+x^4}$.

Proof. Consider $C(x) = \sum_{j=1}^{\infty} z_{2j-1}x^{2j-1}$. The formal power series representation of the generating function for $C(x)$ is

$$C(x) = z_1x + z_3x^3 + \cdots + z_{2r-1}x^{2r-1} + \cdots = \sum_{j=1}^{\infty} z_{2j-1}x^{2j-1}\ .$$

By multiplying this series by $(36ab-2)x^2$, and x^4 respectively, we can get the following series;

$$\begin{aligned}(36ab-2)x^2C(x) &= (36ab-2)z_1x^3 + (36ab-2)z_3x^5 + \cdots + (36ab-2)z_{2r-1}x^{2r+1} + \cdots \\ &= (36ab-2)\sum_{j=1}^{\infty} z_{2j-1}x^{2j+1}\ ,\end{aligned}$$

and

$$x^4C(x) = z_1x^5 + z_3x^7 + \cdots + z_{2r-1}x^{2r+3} + \cdots = \sum_{j=1}^{\infty} z_{2j-1}x^{2j+3}\ .$$

Therefore, we can write

$$\begin{aligned} & \big(1-(36ab-2)x^2+x^4\big)C(x) \\ = \; & z_1x+\big(z_3-(36ab-2)z_1\big)x^3+\big(z_5-(36ab-2)z_3+z_1\big)x^5+\cdots \\ & +\big(z_{2r+1}-(36ab-2)z_{2r-1}+z_{2r-3}\big)x^{2r+1}+\ldots \\ = \; & z_1x+\big(z_3-(36ab-2)z_1\big)x^3 \, . \end{aligned}$$

Since $z_{2n+1}=(36ab-2)z_{2n-1}-z_{2n-3}$, Proposition 1, item (b). And we get

$$C(x)=\frac{z_1x+\big(z_3-(36ab-2)z_1\big)x^3}{1-(36ab-2)x^2+x^4} \, .$$

Now using $z_1=d$ and $z_3=36abd-6bc-d$ we get the result. □

Theorem 2. *The generating function $FG_{z_n}(x)$ for the generalized bi-periodic Lucas-balancing numbers $\{z_n\}_{n\geq 0}$ is*

$$\begin{aligned} FG_{z_n}(x) = & \frac{c+(d-6bc)x+[6a(d-b)+2c(1-18ab)]x^2+(18ab-1)(12bc-2d)x^3}{(1-6bx+x^2)(1-(36ab-2)x^2+x^4)} \\ + & \frac{[6d(a-b)+36(b^2c-ad)]x^4+(d-6bc)x^5}{(1-6bx+x^2)(1-(36ab-2)x^2+x^4)} \, . \end{aligned}$$

Proof. Consider $FG_{z_n}(x)$ the generating function for the generalized bi-periodic Lucas-balancing. The formal power series representation of the generating function for $FG_{z_n}(x)$ is

$$FG_{z_n}(x)=z_0+z_1x+z_2x^2+\cdots+z_rx^r+\cdots=\sum_{j=0}^{\infty}z_jx^j \, .$$

By multiplying this series by $6bx$ and x^2 respectively, we can get the following series;

$$6bxFG_{z_n}(x)=6bz_0x+6bz_1x^2+6bz_2x^3+\cdots+6bz_rx^{r+1}+\cdots=6b\sum_{j=0}^{\infty}z_jx^{j+1} \, ,$$

and

$$x^2FG_{z_n}(x)=z_0x^2+z_1x^3+z_2x^4+\cdots+z_rx^{r+2}+\cdots=\sum_{j=0}^{\infty}z_jx^{j+2} \, .$$

Therefore, we can write

$$\begin{aligned} & (1-6bx+x^2)FG_{z_n}(x) \\ = \;& z_0+(z_1-6bz_0)x+(z_2-6bz_1+z_0)x^2+\cdots+(z_{r+2}-6bz_{r+1}+z_r)x^{r+2}+\cdots \\ = \;& z_0+(z_1-6bz_0)x+\sum_{j=2}^{\infty}(z_j-6bz_{j-1}+z_{j-2})x^j \\ = \;& z_0+(z_1-6bz_0)x+\sum_{j=1}^{\infty}(z_{2j}-6bz_{2j-1}+z_{2j-2})x^{2j}\ . \end{aligned}$$

According to Equation (2), we have $z_{2r+1}=6bz_{2r}-z_{2r-1}$, and $z_{2r}=6az_{2r-1}-z_{2r-2}$, and we get:

$$\begin{aligned} (1-6bx+x^2)FG_{z_n}(x) &= z_0+(z_1-6bz_0)x+\sum_{j=1}^{\infty}(z_{2j}-6bz_{2j-1}+z_{2j-2})x^{2j} \\ &= z_0+(z_1-6bz_0)x+\sum_{j=1}^{\infty}6(a-b)z_{2j-1}x^{2j} \\ &= z_0+(z_1-6bz_0)x+6(a-b)x\sum_{j=1}^{\infty}z_{2j-1}x^{2j-1} \qquad (4) \end{aligned}$$

In Equation (4), we define as $C(x)=\sum_{j=1}^{\infty}z_{2j-1}x^{2j-1}$. Now using the Lemma 2, $z_0=c$, and $z_1=d$, we get that

$$(1-6bx+x^2)FG_{z_n}(x) = c+(d-6bc)x+6(a-b)x\frac{dx+(d-6bc)x^3}{1-(36ab-2)x^2+x^4}\ .$$

Simplifying this, we have the generating function for the generalized bi-periodic Lucas-balancing numbers $FG_{z_n}(x)$. □

4 The Binet formula

In this section, we will provide the Binet formula to the generalized bi-periodic Lucas-balancing numbers. Recall the recurrence (3) with the characteristic polynomial associated to given by

$$z^2-6a^{1-\xi(n)}b^{\xi(n)}z+1=0\ . \qquad (5)$$

Denote the discriminant $\Lambda=(6a^{1-\xi(n)}b^{\xi(n)})^2-4$ and consider $\Lambda>0$. In this case, the roots of the polynomial (5) are simple, and given by $\lambda_1=\frac{6a^{1-\xi(n)}b^{\xi(n)}-\sqrt{\Lambda}}{2}$ and $\lambda_2=\frac{6a^{1-\xi(n)}b^{\xi(n)}+\sqrt{\Lambda}}{2}$. Therefore, for $\Lambda>0$ we can establish the following result.

Proposition 2. For any four non-zero real numbers a, b, c and d, with $6a^{1-\xi(n)}b^{\xi(n)} > 4$, the Binet formula for the generalized bi-periodic Lucas-balancing numbers is given by

$$z_n = \frac{c((\lambda_1)^{n-1} - (\lambda_2)^{n-1}) + d((\lambda_2)^n - (\lambda_1)^n)}{\sqrt{\Lambda}}, \tag{6}$$

where $\lambda_1 = \frac{6a^{1-\xi(n)}b^{\xi(n)} - \sqrt{\Lambda}}{2}$ and $\lambda_2 = \frac{6a^{1-\xi(n)}b^{\xi(n)} + \sqrt{\Lambda}}{2}$.

Proof. Since $\Lambda > 0$, then the roots $\frac{6a^{1-\xi(n)}b^{\xi(n)} + \sqrt{\Lambda}}{2}$ and $\frac{6a^{1-\xi(n)}b^{\xi(n)} - \sqrt{\Lambda}}{2}$ are simple, and the solution of z_n is given by

$$z_n = t_1(\lambda_1)^n + t_2(\lambda_2)^n.$$

Finally, the real numbers t_1 and t_2 that are solutions of the system of equations

$$\begin{cases} t_1 + t_2 = c \\ t_1(\lambda_1)^1 + t_2(\lambda_2)^1 = d \end{cases}.$$

The system is equivalent to the matrix form:

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix},$$

or

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}.$$

Since $\lambda_2 - \lambda_1 = \sqrt{\Lambda}$ and $\lambda_2\lambda_1 = 1$ then $t_1 = \frac{c\lambda_2 - d}{\sqrt{\Lambda}}$ and $t_2 = \frac{d - c\lambda_1}{\sqrt{\Lambda}}$. Therefore, we have

$$z_n = \frac{c\lambda_2 - d}{\sqrt{\Lambda}}(\lambda_1)^n + \frac{d - c\lambda_1}{\sqrt{\Lambda}}(\lambda_2)^n,$$

or equivalently,

$$z_n = \frac{c((\lambda_1)^{n-1} - (\lambda_2)^{n-1}) + d((\lambda_2)^n - (\lambda_1)^n)}{\sqrt{\Lambda}}.$$

□

Now, suppose that $\Lambda = 0$, that is, $6a^{1-\xi(n)}b^{\xi(n)} = 4$. In this case, the root of the polynomial (5) is given by $\lambda = 3a^{1-\xi(n)}b^{\xi(n)}$. By considering this case, we have the following result.

Proposition 3. For any four non-zero real numbers a, b, c and d, with $6a^{1-\xi(n)}b^{\xi(n)} = 4$, the Binet formula for the generalized bi-periodic Lucas-balancing numbers is given by

$$z_n = (c\lambda(1-n) + nd)\lambda^{n-1}, \tag{7}$$

where $\lambda = 3a^{1-\xi(n)}b^{\xi(n)}$.

Proof. Since $\Lambda = 0$, then $\lambda = \frac{6a^{1-\xi(n)}b^{\xi(n)}}{2}$. The solution of z_n is given by

$$z_n = (t_1 + nt_2)\lambda^n$$

where t_1 and t_2 are solutions of the system of equations,

$$\begin{cases} t_1 = c \\ (t_1 + t_2)\lambda^1 = d \end{cases},$$

or, $t_1 = c$ and $t_2 = \frac{d}{\lambda} - c$. Therefore, $z_n = (t_1 + nt_2)\lambda^n = (c + n(\frac{d}{\lambda} - c))\lambda^n = (c\lambda(1-n) + nd)\lambda^{n-1}$. □

5 Some identities

In this section, we will provide some identities for the generalized bi-periodic Lucas-balancing numbers by considering the Binet formulas for the generalized bi-periodic Lucas-balancing numbers.

5.1 The case of $\Lambda > 0$

First, consider the Binet formula with $\Lambda > 0$, namely, Proposition 2. Next, we establish Tagiuri-Vajda?s identity for the generalized bi-periodic Lucas-balancing numbers.

Theorem 3. *For $n, k, r \in \mathbb{Z}$,*

$$z_{n+k}z_{n+r} - z_n z_{n+k+r} = t_1 t_2(\lambda_1^{n+k}\lambda_2^{n+r} + \lambda_1^{n+r}\lambda_2^{n+k} - \lambda_1^{n}\lambda_2^{n+k+r} - \lambda_1^{n+k+r}\lambda_2^{n}) \,. \quad (8)$$

Proof. Applying Binet?s formula, namely, Equation (6) and with straightforward calculations, we obtain

$$\begin{aligned} & z_{n+k}z_{n+r} \\ = \; & (t_1\lambda_1^{n+k} + t_2\lambda_2^{n+k})(t_1(\lambda_1)^{n+r} + t_2(\lambda_2)^{n+r}) \\ = \; & t_1^2\lambda_1^{2n+k+r} + t_2^2\lambda_2^{2n+k+r} + t_1t_2(\lambda_1^{n+k}\lambda_2^{n+r} + \lambda_1^{n+r}\lambda_2^{n+k}) \,, \end{aligned}$$

and

$$\begin{aligned} & z_n z_{n+k+r} \\ = \; & (t_1\lambda_1^{n} + t_2\lambda_2^{n})(t_1(\lambda_1)^{n+k+r} + t_2(\lambda_2)^{n+k+r}) \\ = \; & t_1^2\lambda_1^{2n+k+r} + t_2^2\lambda_2^{2n+k+r} + t_1t_2(\lambda_1^{n}\lambda_2^{n+k+r} + \lambda_1^{n+k+r}\lambda_2^{n}) \,. \end{aligned}$$

Then,

$$\begin{aligned} & z_{n+k}z_{n+r} - z_n z_{n+k+r} \\ = \; & t_1 t_2 (\lambda_1^{n+k}\lambda_2^{n+r} + \lambda_1^{n+r}\lambda_2^{n+k} - \lambda_1^{n}\lambda_2^{n+k+r} - \lambda_1^{n+k+r}\lambda_2^{n}) \,, \end{aligned}$$

which proves Equation (8). □

As consequences of Tagiuri-Vajda?s identity, the last results establish, respectively, d?Ocagne?s identity, Catalan?s identity and Cassini?s identity for the generalized bi-periodic Lucas-balancing numbers.

Proposition 4 (d?Ocagne?s identity). For $m \geq n \in \mathbb{Z}$, then

$$z_{n+1}z_m - z_n z_{m+1} = 0 \,.$$

Proof. Consider $r = m - n$ and $k = 1$ in Equation (8), then

$$\begin{aligned} & z_{n+1}z_m - z_n z_{m+1} \\ = \; & t_1 t_2 (\lambda_1^{n+1}\lambda_2^{m} + \lambda_1^{m}\lambda_2^{n+1} - \lambda_1^{n}\lambda_2^{m+1} - \lambda_1^{m+1}\lambda_2^{n}) \\ = \; & t_1 t_2 \lambda_1^{n}\lambda_2^{m}(\lambda_1 + \lambda_2 - \lambda_2 - \lambda_1) = 0, \end{aligned}$$

which proves the result. □

Proposition 5 (Catalan?s identity). For $n, r \in \mathbb{Z}$ then

$$z_{n+k}z_{n-k} - (z_n)^2 = t_1 t_2 \lambda_1^{n-k}\lambda_2^{n-k}(\lambda_1^k - \lambda_2^k)^2 \,. \tag{9}$$

Proof. Taking $r = -k$ in Equation (8), we have that

$$\begin{aligned} & z_{n+k}z_{n-k} - z_n^2 \\ = \; & t_1 t_2 (\lambda_1^{n+k}\lambda_2^{n-k} + \lambda_1^{n-k}\lambda_2^{n+k} - \lambda_1^{n}\lambda_2^{n} - \lambda_1^{n}\lambda_2^{n}) \\ = \; & t_1 t_2 \lambda_1^{n-k}\lambda_2^{n-k}(\lambda_1^{2k} + \lambda_2^{2k} - 2\lambda_1^{k}\lambda_2^{k}) \\ = \; & t_1 t_2 \lambda_1^{n-k}\lambda_2^{n-k}(\lambda_1^{k} - \lambda_2^{k})^2. \end{aligned}$$

□

As a consequence of Catalan’s identity, by doing $k = 1$ in (9), we have the following result.

Corollary 1 (Cassini?s identity). *For all* $n \in \mathbb{Z}$ *then*

$$z_{n+1}z_{n-1} - z_n^2 = t_1 t_2 \lambda_1^{n-1}\lambda_2^{n-1}(\lambda_1 - \lambda_2)^2 \,.$$

5.2 The case of $\Lambda = 0$

Now consider the Binet formula with $\Lambda = 0$, namely, Proposition 3. Making $A(n) = (c + n(\frac{d}{\lambda} - c)$, we rewrite the Equation (7) and denote the solution by

$$z_n = A(n) \cdot \lambda^n \,. \tag{10}$$

Next, we establish the second Tagiuri-Vajda?s identity for the generalized bi-periodic Lucas-balancing numbers.

Theorem 4. *For $n, k, r \in \mathbb{Z}$,*

$$z_{n+k}z_{n+r} - z_n z_{n+k+r} = \big(A(n+k)A(n+r) - A(n)A(n+k+r)\big)\lambda^{2n+k+r} \,. \tag{11}$$

Proof. Applying Binet?s formula, namely, Equation (10), and by a direct calculation, we get

$$\begin{aligned} & z_{n+k}z_{n+r} - z_n z_{n+k+r} \\ = \; & A(n+k) \cdot \lambda^{n+k} \cdot A(n+r) \cdot \lambda^{n+r} - A(n) \cdot \lambda^n \cdot A(n+k+r) \cdot \lambda^{n+k+r} \\ = \; & \big(A(n+k)A(n+r) - A(n)A(n+k+r)\big)\lambda^{2n+k+r}, \end{aligned}$$

which proves the result. □

As a consequences of Tagiuri-Vajda?s identity 4, the d?Ocagne?s identity, Catalan?s identity and Cassini?s identities for the generalized bi-periodic Lucas-balancing numbers are established next.

Proposition 6 (Second d'Ocagne's identity). For $m \geq n \in \mathbb{Z}$ then

$$z_{n+1}z_m - z_n z_{m+1} = \big(A(n+1)A(m) - A(n)A(m+1)\big)\lambda^{m+n+1} \,.$$

Proof. It suffices to consider $r = m - n$ and $k = 1$ in Equation (11). □

Proposition 7 (Second Catalan?s identity). For $n, r \in \mathbb{Z}$ then

$$z_{n+k}z_{n-k} - (z_n)^2 = \big(A(n+k)A(n-k) - A(n)^2\big)\lambda^{2n} \,.$$

Proof. We get the result by taking $r = -k$ in Equation (11). □

Now, making $k = 1$, we obtain the next result.

Corollary 2 (Second Cassini?s identity)**.** *For all $n \in \mathbb{Z}$ then*

$$z_{n+1}z_{n-1} - z_n^2 = \big(A(n+1)A(n-1) - A(n)^2\big)\lambda^{2n} \,.$$

6 Conclusions

This study has introduced a new generalization of the Lucas-balancing numbers, referred to as the generalized bi-periodic balancing numbers. We derived the generating function for this new family of sequences and established the Binet formulas, as well as several classical identities, including those by Tagiuri-Vajda, d'Ocagne, Catalan, and Cassini. It seems to us that the results presented here are new in the literature.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Ethics Committee Approval

Not applicable.

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