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An alternative method for solving fourth degree polynomial equations

Um método alternativo para resolver equações polinomiais do quarto grau

Un método alternativo para resolver ecuaciones polinomiales de cuarto grado

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Abstract. This article presents an alternative method for solving fourth degree polynomial equations. Although such a result had already been aimed for some time ago, back in the 16th century, by the Italian mathematician Lodovico Ferrari, this work gains originality for being relatively outside of other methods previously discussed. In this work, we will present two original theorems and two corollaries. We will start by introducing a special model of a fourth degree polynomial, which allows us to clearly visualize all its roots. Next, we will demonstrate the main result of this study: the ability to convert any generic fourth-degree polynomial into a special format, thus facilitating the identification of its roots. This method offers a different perspective on solving complex polynomial equations, providing a clear and systematic framework for dealing with problems that have defied conventional methods. Finally, practical examples will be presented that illustrate the application of this method. It is hoped that this result can serve as inspiration and basis for future work that addresses this topic in contemporary mathematics.

Keywords. Polynomial equations, roots of equations, fourth degree polynomial.

Resumo. Este artigo apresenta um método alternativo para resolver equações polinomiais de quarto grau. Embora tal resultado já tivesse sido almejado há algum tempo, ainda no século XVI, pelo matemático italiano Lodovico Ferrari, este trabalho ganha originalidade por estar relativamente fora de outros métodos discutidos anteriormente. Neste trabalho, apresentaremos dois teoremas originais e dois corolários. Começaremos introduzindo um modelo especial de polinômio de quarto grau, que permite visualizar todas as suas raízes de maneira clara. Em seguida, demonstraremos o resultado principal deste estudo: a capacidade de converter qualquer polinômio genérico de quarto grau em um formato especial, facilitando assim a identificação de suas raízes. Este método oferece uma perspectiva diferente na resolução de equações polinomiais complexas, proporcionando uma estrutura clara e sistemática para lidar com problemas que desafiaram métodos convencionais. Por fim, serão apresentados exemplos práticos que ilustram a aplicação deste método. Espera-se que este resultado possa servir de inspiração e base para trabalhos futuros que abordem este tema na matemática contemporânea.

Palavras-chave. Equações polinomiais, raízes de equações, polinômio do quarto grau.

Resumen. Este artículo presenta un método alternativo para resolver ecuaciones polinómicas de cuarto grado. Aunque tal resultado ya había sido perseguido hace algún tiempo, allá por el siglo XVI, por el matemático italiano Lodovico Ferrari, este trabajo gana en originalidad por estar relativamente fuera de otros métodos discutidos anteriormente. En este trabajo presentaremos dos teoremas originales y dos corolarios. Comenzaremos introduciendo un modelo especial de polinomio de cuarto grado, que nos permite visualizar claramente todas sus raíces. A continuación, demostraremos el principal resultado de este estudio: la capacidad de convertir cualquier polinomio genérico de cuarto grado a un formato especial, facilitando así la identificación de sus raíces. Este método ofrece una perspectiva diferente sobre la resolución de ecuaciones polinómicas complejas, proporcionando un marco claro y sistemático para abordar problemas que han desafiado los métodos convencionales. Finalmente, se presentarán ejemplos prácticos que ilustran la aplicación de este método. Se espera que este resultado pueda servir como inspiración y base para futuros trabajos que aborden este tema en las matemáticas contemporáneas.

Palabras-clave. Ecuaciones polinómicas, raíces de ecuaciones, polinomio de cuarto grado.

Mathematics Subject Classification (MSC): 11C08, 12D10.

1 Introduction

Solving polynomial equations has been an intriguing challenge in the history of mathematics since the first studies on the subject. The search for effective and elegant methods to find the roots of these equations has been an area of constant interest. According to [1], in the year 1545 the solution not only to cubic equations, but also to quartic equations, became widely known with the publication of "Ars Magna" by Girolamo Cardano (1501-1576).

However, the search for alternative methods persists, driven by the desire to obtain a in-deeper understanding of the structure of polynomial equations and the search for more accessible and generalized methods. Other approaches and methods have been proposed to solve polynomial equations of the fourth degree, where we can mention, for example, the works of [5], [2] and [6].

When we are working with polynomial equations of order greater than or equal to five, the theory of Évariste Galois (1811-1832) presented impactful results. Through this theory, it was proven that there is no general formula for the roots of a polynomial equation of the fifth degree (or higher) in terms of basic arithmetic operations and root extractions [4].

In this work we will present an alternative method of solving the fourth order polynomial equation, reducing it to second and third degree equations. This work was inspired by [3], who presents this method in a modest way. As an unprecedented result of this work, two theorems and two corollaries will be presented that offer a renewed perspective in approaching these issues.

To begin, we will consider a polynomial with a distinct property (a special polynomial) and illustrate the procedure for finding its roots. At this point, the first theorem will be presented, which allows for the direct identification of a polynomial's roots by analyzing only its coefficients. Next, the main result of this article will be presented: a detailed demonstration that it is always possible to convert a generic polynomial into a special polynomial. Furthermore, it will be shown that this conversion is closed within the field of real numbers. Finally, we will present direct applications of the proposed method, highlighting the simplicity of this approach by applying it to some examples.

2 Special Polynomial

Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial function, of order 4, where $P(x) = x^4 + Kx^3 + Mx^2 + Rx + L$, with $K, M, R, L \in \mathbb{C}$. We will define the polynomial P as special if its coefficients satisfy the following equality.

$$LK^2 = R^2 \tag{1}$$

At this stage, we will demonstrate that it is always possible to find the roots of any special fourth-degree polynomial. This is the main goal of this section.

Consider P a polynomial that satisfies the equation (1). According to the excluded middle principle, either $R = 0$ or $R \neq 0$.

Case 1. Consider $R = 0$.

When $R = 0$ we have two possibilities: $L = 0$ or $K = 0$.

Case 1.1. Consider $L = 0$.

We know that $R = L = 0$. Then the polynomial equation reduces to

$$x^4 + Kx^3 + Mx^2 = 0$$

Putting the term x^2 in evidence, we have

$$x^2(x^2 + Kx + M) = 0$$

Therefore, the quadratic formula yields

$$x_1 = 0,\ \ x_2 = 0,\ \ x_3 = \frac{-K + \sqrt{K^2 - 4M}}{2},\ \ x_4 = \frac{-K - \sqrt{K^2 - 4M}}{2}$$

Case 1.2. Consider $K = 0$.

We know that $R = K = 0$. Then the polynomial equation reduces to

$$x^4 + Mx^2 + L = 0$$

Let $y = x^2$. Therefore,

$$y^2 + My + L = 0$$

.

So, we have

$$y_1 = \frac{-M + \sqrt{M^2 - 4L}}{2} \quad \text{e} \quad y_2 = \frac{-M - \sqrt{M^2 - 4L}}{2}$$

Thus,

$$x_1 = \sqrt{\frac{-M + \sqrt{M^2 - 4L}}{2}},\ \ x_2 = -\sqrt{\frac{-M + \sqrt{M^2 - 4L}}{2}}$$

$$x_3 = \sqrt{\frac{-M - \sqrt{M^2 - 4L}}{2}},\ \ x_4 = -\sqrt{\frac{-M - \sqrt{M^2 - 4L}}{2}}$$

Case 2. Consider $R \neq 0$.

Since the P polynomial is special, we have $LK^2 = R^2$. Furthermore, $R \neq 0$ implies that $L \neq 0$ and $K \neq 0$. As $L \neq 0$, $x = 0$ cannot be a solution to the equation (2)

$$x^4 + Kx^3 + Mx^2 + Rx + L = 0 \tag{2}$$

Therefore, we can divide the (2) equation by x^2 and use $L = \dfrac{R^2}{K^2}$, as follows

$$x^2 + Kx + M + \frac{R}{x} + \frac{L}{x^2} = 0$$

$$x^2 + Kx + M + \frac{R}{x} + \frac{R^2}{K^2x^2} = 0$$

$$\left(x + \frac{R}{Kx}\right)^2 + K\left(x + \frac{R}{Kx}\right) + \left(M - 2\frac{R}{K}\right) = 0$$

Letting $z = x + \dfrac{R}{Kx}$ we obtain,

$$z^2 + Kz + \left(M - 2\frac{R}{K}\right) = 0$$

Employing quadratic formula we see that

$$z_1 = \frac{-K + \sqrt{K^2 - 4\left(M - 2\frac{R}{K}\right)}}{2} \quad \text{and} \quad z_2 = \frac{-K - \sqrt{K^2 - 4\left(M - 2\frac{R}{K}\right)}}{2}$$

Using the definition of z above, we write

$$Kx^2 - (Kz)x + R = 0$$

Finally, we apply the quadratic formula to conclude that

$$x_1 = \frac{Kz_1 + \sqrt{(Kz_1)^2 - 4KR}}{2K}, \quad x_2 = \frac{Kz_1 - \sqrt{(Kz_1)^2 - 4KR}}{2K}$$

$$x_3 = \frac{Kz_2 + \sqrt{(Kz_2)^2 - 4KR}}{2K}, \quad x_4 = \frac{Kz_2 - \sqrt{(Kz_2)^2 - 4KR}}{2K}$$

Now, it is possible to state the first result of this work.

Theorem 1. *Let $P(x) = x^4 + Kx^3 + Mx^2 + Rx + L$ be a special fourth degree polynomial function, with complex coefficients. Then,*

a) If $R = 0$ and $L = 0$, the roots of the polynomial P are:

$$x_1 = 0,\ \ x_2 = 0,\ \ x_3 = \frac{-K+\sqrt{K^2-4M}}{2}\ \ \textit{and}\ \ x_4 = \frac{-K-\sqrt{K^2-4M}}{2}$$

b) If $R = 0$ and $K = 0$, the roots of the polynomial P are:

$$x_1 = \sqrt{\frac{-M+\sqrt{M^2-4L}}{2}},\ \ x_2 = -\sqrt{\frac{-M+\sqrt{M^2-4L}}{2}}$$

$$x_3 = \sqrt{\frac{-M-\sqrt{M^2-4L}}{2}},\ \ x_4 = -\sqrt{\frac{-M-\sqrt{M^2-4L}}{2}}$$

c) If $R \neq 0$, the roots of the polynomial P are:

$$x_1 = \frac{Kz_1+\sqrt{(Kz_1)^2-4KR}}{2K},\ \ x_2 = \frac{Kz_1-\sqrt{(Kz_1)^2-4KR}}{2K}$$

$$x_3 = \frac{Kz_2+\sqrt{(Kz_2)^2-4KR}}{2K},\ \ x_4 = \frac{Kz_2-\sqrt{(Kz_2)^2-4KR}}{2K}$$

where

$$z_1 = \frac{-K+\sqrt{K^2-4\left(M-2\frac{R}{K}\right)}}{2}\ \ \textit{and}\ \ z_2 = \frac{-K-\sqrt{K^2-4\left(M-2\frac{R}{K}\right)}}{2}$$

3 Main Result

In this section we will present the main result of this work. We will show that it is always possible to convert a generic fourth degree polynomial into a special fourth degree polynomial.

Theorem 2. *Let $P(x) = x^4 + Kx^3 + Mx^2 + Rx + L$ be a polynomial of the fourth degree with complex coefficients. Then, there exists $t \in \mathbb{C}$, such that $P_t(x) = (x+t)^4 + K(x+t)^3 + M(x+t)^2 + R(x+t) + L$ is a special polynomial.*

Demonstration: Let $P(x) = x^4 + Kx^3 + Mx^2 + Rx + L$ a fourth degree polynomial with complex coefficients. If $LK^2 = R^2$, then take $t = 0$. Thus, $P_0(x)$ will be a special polynomial. Suppose now that $LK^2 \neq R^2$. We will prove that there exists $t \in \mathbb{C}$ such that $P_t(x)$ is special. To do this, note that,

$$P_t(x) = (x+t)^4 + K(x+t)^3 + M(x+t)^2 + R(x+t) + L$$

$$P_t(x) = x^4 + (4t+K)x^3 + (6t^2+3Kt+M)x^2 + (4t^3+3Kt^2+2Mt+R)x + t^4 + Kt^3 + Mt^2 + Rt + L$$

Therefore, $P_t(x) = x^4 + Ax^3 + Bx^2 + Cx + D$, where

$$\begin{cases} A &= 4t + K \\ B &= 6t^2 + 3Kt + M \\ C &= 4t^3 + 3Kt^2 + 2Mt + R \\ D &= t^4 + Kt^3 + Mt^2 + Rt + L \end{cases}$$

We will show that there is $t_0 \in \mathbb{C}$, such that $DA^2 = C^2$. Thus, on the one hand we will have

$$\begin{aligned} DA^2 &= (t^4 + Kt^3 + Mt^2 + Rt + L)(4t + K)^2 \\ &= (t^4 + Kt^3 + Mt^2 + Rt + L)(16t^2 + 8tK + K^2) \\ &= 16t^6 + 24Kt^5 + (16M + 9K^2)t^4 + (16R + 8KM + K^3)t^3 + (16L + 8KR \\ &\quad +K^2M)t^2 + (K^2R + 8KL)t + K^2L \end{aligned}$$

On the other hand,

$$\begin{aligned} C^2 &= (4t^3 + 3Kt^2 + 2Mt + R)^2 \\ &= 16t^6 + 9K^2t^4 + 4M^2t^2 + R^2 + 24Kt^5 + 16Mt^4 + 8Rt^3 + 12KMt^3 + 6KRt^2 \\ &\quad +4MRt \\ &= 16t^6 + 24Kt^5 + (9K^2 + 16M)t^4 + (8R + 12KM)t^3 + (4M^2 + 6KR)t^2 + R^2 \\ &\quad +4MRt \end{aligned}$$

Therefore, when we force the equality $DA^2 = C^2$, we find

$$(8R + K^3 - 4KM)t^3 + (16L + 2KR + K^2M - 4M^2)t^2 + (K^2R + 8KL - 4MR)t \\ +K^2L - R^2 = 0 \tag{3}$$

Remember, we are assuming $K^2L - R^2 \neq 0$. Thus, to prove that the equation (3) always admits a solution, it is enough to guarantee that

$$8R + K^3 - 4KM \neq 0 \quad \text{or} \quad 16L + 2KR + K^2M - 4M^2 \neq 0 \tag{4}$$

Suppose, for the sake of argument, that

$$8R + K^3 - 4KM = 0 \tag{5}$$

$$16L + 2KR + K^2M - 4M^2 = 0 \tag{6}$$

By (5), we have $R = \dfrac{4KM - K^3}{8}$. Substitute the value of R into the equation (6). Thus,

$$\begin{aligned} 16L + 2K\left(\frac{4KM - K^3}{8}\right) + K^2M - 4M^2 &= 0 \\ 16L + \frac{4K^2M - K^4}{4} + K^2M - 4M^2 &= 0 \\ 64L + 4K^2M - K^4 + 4K^2M - 16M^2 &= 0 \\ 64L + 8K^2M - K^4 - 16M^2 &= 0 \\ 64L - \left(K^2 - 4M\right)^2 &= 0 \\ \left(\frac{K^2 - 4M}{8}\right)^2 &= L \end{aligned} \quad (7)$$

On the one hand, we have $K^2L - R^2 \neq 0$. On the other hand,

$$R = \frac{4KM - K^3}{8} \quad (5) \quad \text{and} \quad \left(\frac{K^2 - 4M}{8}\right)^2 = L \quad (7)$$

We can conclude that,

$$\begin{aligned} K^2L - R^2 &= K^2\left(\frac{K^2 - 4M}{8}\right)^2 - \left(\frac{4KM - K^3}{8}\right)^2 \\ &= K^2\left(\frac{K^2 - 4M}{8}\right)^2 - \left(\frac{4KM - K^3}{8}\right)^2 \\ &= K^2\left(\frac{K^2 - 4M}{8}\right)^2 - K^2\left(\frac{4M - K^2}{8}\right)^2 \\ &= 0 \end{aligned}$$

Absurd! Therefore, the polynomial equation (3) always admits a solution.

□

Corollary 3. *The operation that converts a generic fourth degree polynomial $P(x) = x^4 + Kx^3 + Mx^2 + Rx + L$ into a special fourth degree polynomial $P_{t_0}(x) = x^4 + Ax^3 + Bx^2 + Cx + D$, is closed within the field of real numbers $\mathbb{R}$. That is, let $P(x) = x^4 + Kx^3 + Mx^2 + Rx + L$ be a generic polynomial, with $K, M, R, L \in \mathbb{R}$, then there exists $t_0 \in \mathbb{R}$ such that $P_{t_0}(x) = x^4 + Ax^3 + Bx^2 + Cx + D$ is a special polynomial.*

Demonstration: Let $P(x) = x^4 + Kx^3 + Mx^2 + Rx + L$ a generic fourth polynomial. By the Theorem 2, we know that there is $t_0 \in \mathbb{C}$, such that $P_{t_0}(x)$ is a special fourth degree polynomial. We will prove that, when $K, M, R, L \in \mathbb{R}$, the equation (3) will always admit a real solution. By the excluded middle principle, or $8R + K^3 - 4KM \neq 0$ or $8R + K^3 - 4KM = 0$.

Case 1. Consider $8R + K^3 - 4KM \neq 0$.

Thus, the polynomial equation (3) will be of the third degree, with all its coefficients being real numbers. Therefore, by the Fundamental Theorem of Algebra, it admits at least one real solution.

Case 2. Consider $8R + K^3 - 4KM = 0$.

Thus, the polynomial equation (3) reduces to:

$$Pt^2 + Qt + T = 0 \tag{8}$$

where

$$P = 16L + 2KR + K^2M - 4M^2 \tag{9}$$
$$Q = K^2R + 8KL - 4MR \tag{10}$$
$$R = K^2L - R^2 \tag{11}$$

Substitute $R = \dfrac{4KM - K^3}{8}$ into (9), (10) and (11). Therefore,

$$\begin{aligned} P &= 16L + 2K\left(\frac{4KM - K^3}{8}\right) + K^2M - 4M^2 \\ &= 16L + \frac{(4M - K^2)(2K^2 - 8M)}{8} \\ &= 16L - \frac{(4M - K^2)^2}{4} \\ Q &= K^2R + 8KL - 4MR \\ &= (K^2 - 4M)\left(\frac{4KM - K^3}{8}\right) + 8KL \\ &= 8KL - \frac{K(4M - K^2)^2}{8} \\ T &= K^2L - R^2 \end{aligned}$$

$$= K^2L - \left(\frac{4KM - K^3}{8}\right)^2$$

$$= K^2L - \frac{K^2(4M - K^2)^2}{64}$$

Let $w = (4M - K^2)^2$. Like this,

$$P = 16L - \frac{w}{4}, \quad Q = 8KL - \frac{Kw}{8} \quad \text{and} \quad T = K^2L - \frac{K^2w}{64}$$

The polynomial equation (8) will always have the null discriminant. Indeed,

$$\begin{aligned}\Delta &= Q^2 - 4PT \\ &= \left(8KL - \frac{Kw}{8}\right)^2 - 4\left(16L - \frac{w}{4}\right)\left(K^2L - \frac{K^2w}{64}\right) \\ &= 64K^2L^2 - 2K^2Lw + \frac{K^2w^2}{64} - 64K^2L^2 + K^2Lw + K^2Lw - \frac{K^2w^2}{64} \\ &= 0.\end{aligned}$$

As $8R + K^3 - 4KM = 0$, by (4) we ensure that $P \neq 0$. Since $\Delta = 0$, we have $t_1 = t_2 = \dfrac{-Q}{2P} \in \mathbb{R}$.

□

Corollary 4. *Let $P(x) = x^4 + Kx^3 + Mx^2 + Rx + L$ be a polynomial of the fourth degree with complex coefficients. Then, it is always possible to display its roots using algebraic transformations of its radicals.*

Demonstration: If $P(x)$ is a special polynomial, by the Theorem 1 we already know its roots. If $P(x)$ is not special, take $t_0 \in \mathbb{C}$, such that $P_{t_0}(x) = (x + t_0)^4 + K(x + t_0)^3 + M(x + t_0)^2 + R(x + t_0) + L$ be a special polynomial. By the Theorem 1, we know the roots z_1, z_2, z_3, z_4 of the polynomial $P_{t_0}(x)$. Then the roots of $P(x)$ are

$$x_1 = z_1 + t_0, \ \ x_2 = z_2 + t_0, \ \ x_3 = z_3 + t_0, \ \ x_4 = z_4 + t_0$$

□

4 Applications of the Theorem

Example 1. *Find the roots of the polynomial* $P(x) = x^4 + x^3 + x^2 + x + 1$.

Demonstration: Note that, $K = 1, M = 1, R = 1, L = 1$. Thus, $LK^2 = R^2$. Therefore the polynomial $P(x)$ is special. As $K \neq 0$, we have:

$$z_1 = \frac{-K + \sqrt{K^2 - 4\left(M - 2\frac{R}{K}\right)}}{2} = \frac{-1 + \sqrt{(1)^2 - 4\left(1 - 2\frac{1}{1}\right)}}{2} = \frac{-1 + \sqrt{5}}{2}$$

$$z_2 = \frac{-K - \sqrt{K^2 - 4\left(M - 2\frac{R}{K}\right)}}{2} = \frac{-1 - \sqrt{(1)^2 - 4\left(1 - 2\frac{1}{1}\right)}}{2} = \frac{-1 - \sqrt{5}}{2}$$

Like this,

$$x_1 = \frac{Kz_1 + \sqrt{(Kz_1)^2 - 4KR}}{2K} = \frac{\frac{-1+\sqrt{5}}{2} + \sqrt{\frac{3-\sqrt{5}}{2} - 4}}{2} = \frac{-1 + \sqrt{5}}{4} + \frac{\sqrt{10 + 2\sqrt{5}}}{4}i$$

$$x_2 = \frac{Kz_1 - \sqrt{(Kz_1)^2 - 4KR}}{2K} = \frac{\frac{-1+\sqrt{5}}{2} - \sqrt{\frac{3-\sqrt{5}}{2} - 4}}{2} = \frac{-1 + \sqrt{5}}{4} - \frac{\sqrt{10 + 2\sqrt{5}}}{4}i$$

$$x_3 = \frac{Kz_2 + \sqrt{(Kz_2)^2 - 4KR}}{2K} = \frac{\frac{-1-\sqrt{5}}{2} + \sqrt{\frac{3+\sqrt{5}}{2} - 4}}{2} = \frac{-1 - \sqrt{5}}{4} + \frac{\sqrt{10 - 2\sqrt{5}}}{4}i$$

$$x_4 = \frac{Kz_2 - \sqrt{(Kz_2)^2 - 4KR}}{2K} = \frac{\frac{-1-\sqrt{5}}{2} - \sqrt{\frac{3+\sqrt{5}}{2} - 4}}{2} = \frac{-1 - \sqrt{5}}{4} - \frac{\sqrt{10 - 2\sqrt{5}}}{4}i$$

□

Example 2. *Find the roots of the polynomial* $P(x) = x^4 + 2x^3 + 2x^2 + x + 1$.

Demonstration: Note that, $K = 2, M = 2, R = 1, L = 1$. Thus, $LK^2 \neq R^2$. Therefore, $P(x)$ is not a special polynomial. We will find $t_0 \in \mathbb{R}$, so that

$$P_{t_0}(z) = (z + t_0)^4 + 2(z + t_0)^3 + 2(z + t_0)^2 + (z + t_0) + 1$$

be a special polynomial. Remember that t_0 is a real root of the polynomial equation (3). So, we have

$$12t^2 + 12t + 3 = 0 \quad \Rightarrow \quad t_0 = -\frac{1}{2}$$

Therefore, $P_{t_0}(z) = z^4 + Az^3 + Bz^2 + Cz + D$, where

$$\begin{cases} A &= 4t+K &= 0 \\ B &= 6t^2+3Kt+M &= 1/2 \\ C &= 4t^3+3Kt^2+2Mt+R &= 0 \\ D &= t^4+Kt^3+Mt^2+Rt+L &= 13/16 \end{cases}$$

Then $P_{t_0}(z) = z^4 + \dfrac{z^2}{2} + \dfrac{13}{16}$, with $A = C = 0$, $B = \dfrac{1}{2}$ and $D = \dfrac{13}{16}$. Like this,

$$z_1 = \sqrt{\frac{-B+\sqrt{B^2-4D}}{2}} = \sqrt{-\frac{1}{4}+\frac{\sqrt{3}}{2}i}$$

$$z_2 = -\sqrt{\frac{-B+\sqrt{B^2-4D}}{2}} = -\sqrt{-\frac{1}{4}+\frac{\sqrt{3}}{2}i}$$

$$z_3 = \sqrt{\frac{-B-\sqrt{B^2-4D}}{2}} = \sqrt{-\frac{1}{4}-\frac{\sqrt{3}}{2}i}$$

$$z_4 = -\sqrt{\frac{-B-\sqrt{B^2-4D}}{2}} = -\sqrt{-\frac{1}{4}-\frac{\sqrt{3}}{2}i}$$

Without loss of generality, we can consider

$$\sqrt{-\frac{1}{4}+\frac{\sqrt{3}}{2}i} = \frac{\sqrt{2\sqrt{13}-2}}{4} + \frac{\sqrt{2\sqrt{13}+2}}{4}i$$

$$\sqrt{-\frac{1}{4}-\frac{\sqrt{3}}{2}i} = \frac{\sqrt{2\sqrt{13}-2}}{4} - \frac{\sqrt{2\sqrt{13}+2}}{4}i$$

Therefore, the polynomial roots $P(x) = x^4 + 2x^3 + 2x^2 + x + 1$ are

$$x_1 = -\frac{1}{2} + \frac{\sqrt{2\sqrt{13}-2}}{4} + \frac{\sqrt{2\sqrt{13}+2}}{4}i$$

$$x_2 = -\frac{1}{2} - \frac{\sqrt{2\sqrt{13}-2}}{4} - \frac{\sqrt{2\sqrt{13}+2}}{4}i$$

$$x_3 = -\frac{1}{2} + \frac{\sqrt{2\sqrt{13}-2}}{4} - \frac{\sqrt{2\sqrt{13}+2}}{4}i$$

$$x_4 = -\frac{1}{2} - \frac{\sqrt{2\sqrt{13}-2}}{4} + \frac{\sqrt{2\sqrt{13}+2}}{4}i$$

□

5 Final considerations

Studying the roots of a polynomial equation contributes to understanding the behavior of polynomials. The development of alternative algebraic methods for solving polynomial equations contributes to a deeper understanding of the underlying algebraic structure. This promotes the ability to manipulate algebraic expressions and understand the properties of the roots of polynomials.

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