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On repunit polynomials sequence

Sobre a sequência de polinômios repunidade

Sobre la secuencia de polinomios repunidad

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Abstract. In this paper, we define a sequence of polynomials associated with the repunit numerical sequence. This involves extending the concept of the repunit sequence, a sequence type Horadam, represented by the sequence of repunit numbers, where r_n represents the n-th repunit, and the recurrence relation: $r_{n+1} = 11r_n - 10r_{n-1}$, with $r_0 = 0$, $r_1 = 1$ for $n \geq 1$. In this paper, we investigate this new polynomial sequence in detail and present some results and applications derived from this investigation. For instance, we study the characteristic equation and derive the corresponding generating function. Additionally, we analyze the recurrence relation associated with the sum of n repunit polynomials.

Keywords. Polynomial repunit, repunit number, sequence.

Resumo. Neste trabalho, definimos uma sequência de polinômios associada à sequência numérica repunidade. Isto envolve estender o conceito de sequência repunidade, uma sequência do tipo Horadam, representada pela sequência de números repunidades, em que r_n representa o n-ésima repunidade, e a relação de recorrência: $r_{n+1} = 11r_n - 10r_{n-1}$, com $r_0 = 0$, $r_1 = 1$ para $n \geq 1$. Neste artigo, investigamos esta nova sequência polinomial em detalhe e apresentamos alguns resultados e aplicações derivados desta investigação. Por exemplo, estudamos a equação caraterística e derivamos a função geradora correspondente. Adicionalmente, analisamos a relação de recorrência associada à soma de n polinômios repunidades.

Palavras-chave. Número repunidade, polinômio repunidade, sequência.

Resumem. En este trabajo definimos una sucesión de polinomios asociada a la sucesión de números de repunidades. Se trata de extender el concepto de sucesión de repunidades, una sucesión de tipo Horadam, representada por la sucesión de números de repunidades, donde r_n representa el n-ésimo número de repunidades, y la relación de recurrencia $r_{n+1} = 11r_n - 10r_{n-1}$, con $r_0 = 0$, $r_1 = 1$ para $n \geq 1$. En este artículo, investigamos en detalle esta nueva secuencia polinómica y presentamos algunos resultados y aplicaciones derivados de esta investigación. Por ejemplo, estudiamos la ecuación característica y derivamos la función generatriz correspondiente. Además, analizamos la relación de recurrencia asociada a la suma de n polinomios repunit.

Palabras-clave. Número de repunidad, repunidad polinòmica, secuencia.

Mathematics Subject Classification (MSC): primary 11B37; secondary 11B39, 11B83.

1 Introduction

The *repunit* numbers are defined as r_n, where n is a non-negative integer or natural number ($n \in \mathbb{N}$), a repunit is characterized by its representation in a positional number system with base $b > 1$, consisting solely of the repetition of the digit 1. For instance, in the decimal system (base $b = 10$), examples of repunit numbers include $1, 11, 111, 1111,$ and 11111. These sequences, cataloged as $A002275$ in the OEIS [16], have captured the fascination of mathematicians for generations [1, 3, 4, 8, 18], and beyond. In this paper, we consider numbers r_n of the form $111\ldots1$ $(n$ times $1's)$ when expressed in base 10. Such a number is of the form $r_n = \frac{10^n-1}{9}$.

We start with the elementary observation that the sequence r_n satisfies the linear recurrence

$$r_{n+1} = 10r_n + 1, \text{ with } r_1 = 1 \text{ and } n \geq 1\ . \tag{1}$$

Admitting $r_0 = 0$ we have in the Equation (1) that

$$r_n = 10r_{n-1} + 1, \text{ with } r_0 = 0 \text{ and } n \geq 1\ . \tag{2}$$

Making a difference (1)- (2) we get

$$r_{n+1} = 11r_n - 10r_{n-1} \text{ with } r_0 = 0\ , r_1 = 1, \text{ and } n \geq 1\ , \tag{3}$$

where r_n denotes the n-th repunit, for convenience we use $r_0 = 0$. In [11, 12] the repunit sequence $\{r_n\}_{n\geq 0}$ is recursively defined by the homogeneous relation (3) .

The Horadam-type sequence is a specific class of sequences, defined by the following characteristics: $h_{n+2} = ph_{n+1} + qh_n$, with initial conditions $h_0 = a$, $h_1 = b$ and

p, q real numbers. This sequence like was introduced, in 1965, by Horadam [6], and it generalizes many sequences with characteristic equation of recurrence relation of form $x^2 - px - q = 0$. Thus repunit sequence is a particular case of the Horadam sequence. Recursive relations define several families of integers that are studied in the literature. These sequences of numbers are at the origin of many interesting identities.

Several researchers associate specific numerical sequences with a sequence of polynomials, another way of studying and obtaining results from these sequences and applications, see [5, 13, 14, 15] and associated references. Motivated by these works, In Section 2, we present some preliminary results about the repunit sequence, such as the Binet formula, some classical identities, and generating function for the repunit numbers used in the next sections. In Section 3, we define the *repunit polynomial sequence* associated with the repunit sequence. We study the characteristic equation, the main result being its generating function. Finally, in Section 4, we show another recurrence associated with the sum of n repunit polynomials sequence, in addition to observing the non-convergence of this sequence. Emphasis that repunit polynomial sequence is also a polynomial sequence of the Horadam sequence type. More general work on the polynomials associated with the Horadam sequence was carried out in [7] and [17].

2 Background and preliminaries results

We remember that a recurrence is an expression that defines an element of a sequence based on previously given or known terms. In this section, we present an expression that provides the terms of the repunit sequence $\{r_n\}_{n\geq 0}$ in function exclusively of n and no longer recursively through the previous elements, and a generating function of the repunit sequence. To do this, we will determine the sequences that are solutions of the recurrence $a_n = 11a_{n-1} - 10a_{n-2}$, and among these solutions which satisfy the case in which $a_0 = 0$ and $a_1 = 1$.

In particular, Equation (3) is a linear difference equation, or linear recurrence, of order 2. According [10], if the equation $r^2 + pr + q = 0$ has distinct roots r_1 and r_2, and then the sequences $a_n = c_1(r_1)^n + c_2(r_2)^n$, where $n \in \mathbb{N}$, and c_1, $c_2 \in \mathbb{R}$, are solutions of

$$x_{n+2} + px_{n+1} + qx_n = 0, \text{ for } n \in \mathbb{N}, n \geq 1 .$$

Note that the difference equation associated with the sequence of repunit r_n is

$$r_{n+1} = 11r_n - 10r_{n-1},$$

which has as its characteristic equation $t^2 - 11t + 10 = 0$ and its real roots are $t_1 = 10$

and $t_2 = 1$. We have that a general solution to Equation (4) is of the form

$$r_n = c_1(10)^n + c_2(1)^n \ .$$

Let us determine the constants c_1 and c_2, considering that $r_0 = 0$ and $r_1 = 1$, and we obtain the system,

$$\begin{cases} 0 = c_1 + c_2 \\ 1 = 10c_1 + c_2 \ . \end{cases}$$

Solving the system we find $c_1 = \dfrac{1}{9}$ and $c_2 = -\dfrac{1}{9}$. So we have just shown that:

Proposition 1. *[1, 18] For all $n \in \mathbb{N}$, we have*

$$r_n = \frac{10^n - 1}{9} \ , \tag{4}$$

where $\{r_n\}_{n\geq 0}$ is the repunit sequence.

The Equation (4) in Proposition 1, presents the classic and well-known Binet formula for the sequence of repunit $\{r_n\}_{n\geq 0}$. The Binet formula is a powerful tool for understanding sequences, as it allows us to derive key properties and even determine the generating function of a sequence. By applying the Binet formula to a sequence defined by a recurrence relation, we can directly compute specific terms without needing to calculate each preceding term. This not only aids in understanding the behavior and growth patterns of the sequence but also enables us to analyze its properties more efficiently.

Remember that the repunit sequence $\{r_n\}_{n\geq 0}$ is recursively presented in base 10 by linear recurrence Equation (3). The next results in this section can also be consulted in this reference.

Considering the sequence of partial sums $s_n = r_1 + r_2 + \cdots + r_n$, for $n \geq 1$, where $\{r_n\}_{n\geq 0}$ is the repunit sequence.

Proposition 2. *[11] Let $\{r_n\}_{n\geq 0}$ be the repunit sequence, then*

$$s_n = \frac{10(10^n - 1) - 9n}{81} \ ,$$

where $\{s_n\}_{n\geq 0}$ partial sum of the elements of the $\{r_n\}_{n\geq 0}$.

Now the classical identity:

Proposition 3. *[11] [Catalan's Identity] Let m, n be any natural numbers. For $m \geq n$ we have*

$$r_m^2 - r_{m-n}r_{m+n} \ = \ 10^{m-n} \cdot (r_n)^2 \ ,$$

where $\{r_n\}_{n\geq 0}$ is the repunit sequence.

A direct consequence from Proposition 3.

Corollary 1. *[11] [Cassini's Identity] For all $m \geq 1$, we have $r_m^2 - r_{m-1}r_{m+1} = 10^{m-1}$, where $\{r_n\}_{n\geq 0}$ is the the repunit sequence.*

According [10], the exponential generating function $f(x)$ of a sequence $(a_n)_{n\geq 0}$ is a power series of the form

$$f(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + ... + \frac{a_n x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \ .$$

In the next result, we consider $a_n = r_n$ and obtain the exponential generating function for the repunit sequence $\{r_n\}_{n\geq 0}$.

Proposition 4. *[4] For all $n \geq 0$ the exponential generating function for the repunit sequence $\{r_n\}_{n\geq 0}$ is*

$$\sum_{n=0}^{\infty} \frac{r_n t^n}{n!} = \frac{e^{10t} - e^t}{9} \ .$$

3 The repunit polynomial

Remember that the repunit polynomial sequence also falls under the category of polynomial sequences Horadam type. Existing literature has already explored the concept of generalized Horadam polynomial sequences, as seen in works [7, 17]. Kocer and Horzum [7] in their research, Binet Formula, and Catalan and Cassini identities have been established for generalized polynomial Horadam sequences. Although our proof focuses on the specific polynomial sequence of repunit, it differs notably in terms of the roots of the characteristic polynomial. Additionally, the cited work [7] discusses these identities in relation to coefficients a, b, p, and q. Furthermore, Soykan [17] delves into broader scenarios, exploring generalized Horadam-Leonardo and Horadam-Leonardo-Lucas polynomials. This examination underscores the interconnections among identities involving Horadam, Horadam-Leonardo, and Horadam-Leonardo-Lucas polynomials.

First, we present the definition of repunit polynomials and follow some properties.

Definition 1. *For all $n \geq 1$, the polynomials is given by:*

$$R_1(x) = 1 \ , \ R_2(x) = x + 10 \ \ \textit{and} \ \ R_{n+2}(x) = 11x R_{n+1}(x) - 10 R_n(x) \ , \qquad (5)$$

they are called repunit polynomials, with x being a real variable.

Note that, for $x = 1$, the recurrence given in Equation (5) generates the sequence of repunit numbers $\{r_n\}_{n\geq 0}$. Hence, we also say that these polynomials are associated with the repunit sequence $\{r_n\}_{n\geq 0}$, a Horadam-type sequence.

In the Table (1) we present the repunit polynomials for $1 \leq n \leq 5$.

Table 1: Repunit polynomials

n	$R_n(x)$
1	1
2	$x + 10$
3	$11x^2 + 110x - 10$
4	$121x^3 + 1210x^2 - 120x - 100$
5	$1331x^4 + 13310x^3 - 1430x^2 - 2200x + 100$

Note that the recurrence Equation (5) has a characteristic equation given by

$$Y^2 - 11xY + 10 = 0\ , \tag{6}$$

whose roots are, in the unknown Y:

$$\alpha(x) = \frac{11x + \sqrt{(11x)^2 - 40}}{2} \quad \text{and} \quad \beta(x) = \frac{11x - \sqrt{(11x)^2 - 40}}{2}\ . \tag{7}$$

Note that $\alpha(x) + \beta(x) = 11x$ and $\alpha(x) \cdot \beta(x) = 10$.

With this, we demonstrate that,

Proposition 5. *The characteristic equation* (6) *associated with the recurrence*

$$R_{n+2}(x) - 11xR_{n+1}(x) + 10R_n(x) = 0\ , \tag{8}$$

has as roots $\alpha(x)$ and $\beta(x)$ is given by Equation (7)*, and $\{R_n\}_{n\geq 0}$ is the repunit polynomial sequence.*

The Theorems 1 and 2 presented below, respectively, guarantee the existence and characterize the roots (solutions) of the recurrence repunit polynomial is given by Equation (5). It should be noted that $C_1(x)$ and $C_2(x)$ used in the following results are given in function of x, for simplicity we will use the notation $C_1 = C_1(x)$ and $C_2 = C_2(x)$. As long as there is no confusion.

Theorem 1. *If the distinct roots of the characteristic Equation* (6) *are $\alpha(x)$ and $\beta(x)$, then*

$$Z_n = C_1\alpha^n + C_2\beta^n$$

is the solution of the equation from the recurrence relation 8, whatever the values C_1 and C_2.

Proof. Replacing $Z_n = C_1\alpha^n + C_2\beta^n$ in the recurrence Equation (5), after grouping the terms, we have

$$C_1\alpha^n \underbrace{(\alpha^2 - 11x\alpha + 10)}_{=0} + C_2\beta^n \underbrace{(\beta^2 - 11x\beta + 10)}_{=0} = C_1\alpha^n 0 + C_2\beta^n 0 = 0.$$

Therefore, $Z_n = C_1\alpha^n + C_2\beta^n$ is the solution of the recurrence Equation 8. □

Theorem 2. *If the distinct roots of Equation* (6) *are α and β, then all solutions of the recurrence Equation* (5) *are of the form*

$$Z_n = C_1\alpha^n + C_2\beta^n \ . \tag{9}$$

Proof. Let T_n be any solution of the recurrence Equation (5). Let us determine the functions C_1 and C_2 that are solutions of the system of equations,

$$\begin{cases} C_1\alpha + C_2\beta = 1 \\ C_1\alpha^2 + C_2\beta^2 = x + 10 \end{cases}$$

that is,

$$C_1 = \frac{-\beta + (x+10)}{\alpha(\alpha - \beta)}, \qquad C_2 = \frac{\alpha - (x+10)}{\beta(\alpha - \beta)}. \tag{10}$$

We can state that $T_n = C_1\alpha^n + C_2\beta^n$ for all natural number n, which will show the desired result. In effect, let $A_n = T_n - C_1\alpha^n - C_2\beta^n$. Our interest is to show that $A_n = 0$ for all n. There is,

$$\begin{aligned} &A_{n+2} - 11xA_{n+1} + 10A_n = \\ &(T_{n+2} - 11xT_{n+1} + 10T_n) - C_1\alpha^n(\alpha^2 - 11x\alpha + 10) - C_2\beta^n(\beta^2 - 11x\beta + 10). \end{aligned}$$

The first parenthesis is equal to zero because T_n is a solution of recurrence Equation (5), the last two parentheses are equal to zero because α and β are roots of Equation (6). Then $A_{n+2} - 11xA_{n+1} + 10A_n = 0$. And yet, as $C_1\alpha + C_2\beta = 1$ and $C_1\alpha^2 + C_2\beta^2 = x + 10$, we have $A_1 = A_2 = 0$. But, if $A_{n+2} - 11xA_{n+1} + 10A_n = 0$ and $A_1 = A_2 = 0$, then $A_n = 0$ for all n. □

It should be noted that the Theorem 2 is particular case for the repunit polynomials Horadam-type sequence of Theorem 1 in [10].

In a similar way to Proposition 4, the next result will display the generating function of the repunit polynomial sequence, namely:

Theorem 3. *For all $n \geq 0$, the exponential generating function for the repunit polynomial sequence is*

$$\sum_{n=0}^{\infty} \frac{R_n t^n}{n!} = C_1 e^{\alpha t} + C_2 e^{\beta t} \ ,$$

where $\{R_n\}_{n\geq 0}$ is the repunit polynomial sequence, and α and β are distinct roots of Equation (6)*.*

Proof. The exponential generating function for the repunit polynomial sequence is $\sum_{n=0}^{\infty} \frac{R_n t^n}{n!}$. Making use of Equation (9), we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{R_n t^n}{n!} &= \sum_{n=0}^{\infty} (C_1\alpha^n + C_2\beta^n)\frac{t^n}{n!} \\ &= C_1 \sum_{n=0}^{\infty} \frac{\alpha^n t^n}{n!} + C_2 \sum_{n=0}^{\infty} \frac{\beta^n t^n}{n!} \\ &= C_1 e^{\alpha t} + C_2 e^{\beta t} \ ,\end{aligned}$$

where C_1, C_2, α and β are given in Theorems 1 e 2. □

In the Theorem 2, note that in the particular case where $x = \frac{2\sqrt{10}}{11}$, we have $\alpha = \beta = \sqrt{10}$. In this case, a solution of recurrence Equation (5) is given by,

$$Z_n = c_1 10^{n/2} + c_2 n 10^{n/2}, \tag{11}$$

where c_1 and c_2 are constants. In fact, just replace the expression (11) in the recurrence recurrence Equation (5), obtaining

$$\begin{aligned}&c_1 10^{\frac{n+2}{2}} + c_2(n+2)10^{\frac{n+2}{2}} - 11x(c_1 10^{\frac{n+1}{2}} + c_2(n+1)10^{\frac{n+1}{2}}) + 10(c_1 10^{\frac{n}{2}} + c_2(n)10^{\frac{n}{2}}) \\ &= 10^{\frac{n}{2}} c_1 \underbrace{(10 - 11x10^{\frac{1}{2}} + 10)}_{=0} + 10^{\frac{n}{2}} c_2((n+2)10 - 11x(n+1)10^{\frac{1}{2}} + 10n) \\ &= 10^{\frac{n}{2}} c_2 n \underbrace{(10 - 11x10^{\frac{1}{2}} + 10)}_{=0} + 10^{\frac{n}{2}} c_2 \underbrace{(20 - 11x10^{\frac{1}{2}})}_{=0} = 0.\end{aligned}$$

This shows that $Z_n = c_1 10^{n/2} + c_2 n 10^{n/2}$ is the solution of the recurrence Equation (5).

The next result guarantees that the solutions of the recurrence Equation (5) are always of the form $Z_n = c_1 10^{n/2} + c_2 n 10^{n/2}$, in the case where $\alpha = \beta$. The demonstration of the Theorem 4 will be omitted, but it is analogous to the demonstration of the Theorem 2, for:

$$c_1 = \frac{2\sqrt{10} - (x+10)}{10}, \qquad c_2 = \frac{-\sqrt{10} + (x+10)}{10}, \tag{12}$$

with $x = \frac{2\sqrt{10}}{11}$.

Theorem 4. *Let α and β be the roots of Equation* (6). *If $\alpha = \beta = \sqrt{10}$, then all solutions of the recurrence Equation* (5) *are of the form*

$$Z_n = c_1 10^{n/2} + c_2 10^{n/2} \ , \tag{13}$$

with c_1 and c_2 constant, and particular $x = \frac{2\sqrt{10}}{11}$.

It is important to note that Theorem 4 applies specifically to the repunit polynomials in the Horadam-type sequence from Theorem 2 in [10].

Note that, if α and β are roots of Equation (6), we have

$$\alpha^2 = 11x\alpha - 10 \quad \text{and} \quad \beta^2 = 11x\beta - 10. \tag{14}$$

Multiplying the above expressions by α^{n-1} and β^{n-1}, respectively, we obtain

$$\alpha^{n+1} = 11x\alpha^n - 10\alpha^{n-1} \quad \text{and} \quad \beta^{n+1} = 11x\beta^n - 10\beta^{n-1}. \tag{15}$$

Therefore,

$$\begin{aligned}
\alpha^2 &= 11x\alpha - 10 = P_2(x)\alpha - 10P_1(x) \\
\alpha^3 &= 11x\alpha^2 - 10\alpha = 11x(11x\alpha - 10) - 10\alpha = (\underbrace{(11x)^2 - 10}_{P_3(x)})\alpha - 10(\underbrace{11x}_{P_2(x)}) \\
\alpha^4 &= 11x\alpha^3 - 10\alpha^2 = (\underbrace{(11x)^3 - 220x}_{P_4(x)})\alpha - 10(\underbrace{(11x)^2 - 10}_{P_3(x)}) \\
\alpha^5 &= (\underbrace{(11x)^4 - 3 \cdot 10 \cdot (11x)^2 + 10^2}_{P_5(x)})\alpha - 10(\underbrace{(11x)^3 - 220x}_{P_4(x)}) \\
&\cdots \\
\alpha^n &= 11x\alpha^{n-1} - 10\alpha^{n-2} = P_n(x)\alpha - 10P_{n-1}(x).
\end{aligned}$$

Then we have the following result:

Proposition 6. *Let $\alpha(x)$ and $\beta(x)$ be the roots of Equation* (6), *with $\alpha \neq \beta$, for $n \geq 2$, then $\big(\alpha(x)\big)^n = (P_n(x))\alpha(x) - 10 \cdot P_{n-1}(x)$, with $P_1(x) = 1$ and*

$$P_n(x) = (11x)^{n-1} + \sum_{j=2}^{n} (-10)^{\frac{j}{2}} \big(n - j)\big)(11x)^{n-j} \textit{ for } n \textit{ and } j \textit{ even} \ ;$$

$$P_n(x) = (11x)^{n-1} + \sum_{j=3}^{n} (-10)^{\frac{j-1}{2}} \big(n - (j-1)\big)(11x)^{n-j} \textit{ for } n \textit{ and } j \textit{ odd }.$$

Note that the Proposition 6 also applies to

$$\big(\beta(x)\big)^n = P_n(x)\beta(x) - 10 \cdot P_{n-1}(x). \tag{16}$$

Therefore, we can write

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = P_n(x), \text{ for all natural } n. \tag{17}$$

According to the Theorem 2, the solutions of the recurrence Equation (5) are of the form given in Equation (13), with C_1 and C_2 constant. Therefore, we can rewrite the recurrence solution as follows:

$$\begin{aligned} R_n &= C_1\alpha^n + C_2\beta^n && (18)\\ &= \frac{-\beta + (x+10)}{\alpha(\alpha-\beta)} \cdot \alpha^n + \frac{\alpha - (x+10)}{\beta(\alpha-\beta)} \cdot \beta^n \\ &= \frac{-\alpha^{n-1}\beta + (\alpha^{n-1} - \beta^{n-1})(x+10) + \beta^{n-1}\alpha}{\alpha-\beta} \\ &= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha-\beta} \cdot (x+10) - \alpha\beta \cdot \frac{\alpha^{n-2} - \beta^{n-2}}{\alpha-\beta} \\ &= P_{n-1}(x) \cdot (x+10) - 10 \cdot P_{n-2}(x). && (19) \end{aligned}$$

Also note that each polynomial $P_n(x)$ can be given recursively, since

$$\begin{aligned} \alpha^{n+2} &= 11x\alpha^{n+1} - 10\alpha^n && (20)\\ \beta^{n+2} &= 11x\beta^{n+1} - 10\beta^n. && (21) \end{aligned}$$

Subtracting the equalities (20) and (21) and subsequently dividing by $(\alpha - \beta)$, we obtain

$$\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha-\beta} = 11x \cdot \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta} - 10 \cdot \frac{\alpha^n - \beta^n}{\alpha-\beta}.$$

Which guarantees the following result:

Proposition 7. *For every $n \geq 1$ natural number, we have that*

$$P_{n+2}(x) = 11x \cdot P_{n+1}(x) - 10 \cdot P_n(x),$$

with $P_1(x) = 1$ and $P_2(x) = 11x$.

Let us now look at some classic identities.

Theorem 5. *[Catalan's identity] Let m, n be any natural numbers. For $m \geq n$ we have*

$$(R_m)^2 - R_{m-n} \cdot R_{m+n} = C_1 C_2 \cdot 10^{m-n} \left[2 \cdot 10^n - (\alpha^{2n} + \beta^{2n})\right] \ ,$$

where $\{R_n\}_{n\geq 0}$ is the repunit polynomial sequence, and α and β are distinct roots of Equation (6).

Proof. Using Theorem 2, we have

$$\begin{aligned}(R_m)^2 \ &- \ R_{m-n} \cdot R_{m+n} = (C_1\alpha^m + C_2\beta^m)^2 - \\ &- \ (C_1\alpha^{m-n} + C_2\beta^{m-n})(C_1\alpha^{m+n} + C_2\beta^{m+n}) \\ &= \ C_1^2\alpha^{2m} + 2 \cdot 10^m \cdot C_1C_2 + C_2^2\beta^{2m} + \\ &- \ \left(C_1^2\alpha^{2m} + C_1C_2 \cdot 10^{m-n} \cdot (\alpha^{2n} + \beta^{2n}) + C_2^2\beta^{2m}\right) \\ &= \ C_1C_2 \cdot 10^{m-n} \cdot \left[2 \cdot 10^n - (\alpha^{2n} + \beta^{2n})\right] \ ,\end{aligned}$$

as required. □

Making $n = 1$, follows directly from Theorem 5 that:

Corollary 2. *[11] [Cassini's Identity] For all $m \geq 1$, we have*

$$(R_m)^2 - R_{m+1}R_{m-1} = C_1C_2 \cdot 10^{m-1} \left[20 - (\alpha^2 + \beta^2)\right] \ ,$$

where $\{R_n\}_{n\geq 0}$ is the repunit polynomial sequence, and α and β are distinct roots of Equation (6).

4 Sum of terms involving the polynomial sequence

Here the polynomial $S_n := S_n(x)$ represents the partial sum of n terms of $R_n(x)$, that is, $S_n = R_1 + R_2 + \cdots + R_n = \sum_{i=1}^{n} R_i$. We did not find a sum of terms involving the polynomial sequence in the literature we consulted for generalized polynomial Horadam sequences.

We will show that S_n can also be given by a recurrence.

Proposition 8. *Consider $S_1 = 1$, $S_2 = 11 + x$ and for all $n \geq 3$, we have*

$$S_n = \frac{R_{n+1} - 10R_n + 10x - 11}{11x - 11} \ , \tag{22}$$

where $\{R_n\}_{n\geq 0}$ *is the repunit polynomial sequence, and* $\{S_n\}_{n\geq 0}$ *partial sum of the elements of* $\{R_n\}_{n\geq 0}$.

Proof. Note that,

$$\begin{aligned} R_1 &= 1 \\ R_2 &= 10 + x \\ R_3 &= 11xR_2 - 10R_1 \\ &\cdots \\ R_n &= 11xR_{n-1} - 10R_{n-2}. \end{aligned}$$

Adding the above equalities, we have

$$\begin{aligned} S_n &= R_1 + R_2 + \cdots + R_n \\ &= 1 + (x+10) + (11xR_2 - 10R_1) + \cdots + (11xR_{n-1} - 10R_{n-2}) \\ &= (11+x) + 11x(R_2 + R_3 + \cdots + R_{n-1}) - 10(R_1 + R_2 + \cdots + R_{n-2}) \\ &= 11x(R_1 + R_2 + \cdots + R_{n-1} + R_n) - 10(R_1 + \cdots + R_{n-2} + R_{n-1} + R_n) \\ &\quad +(11+x) - 11x(R_1 + R_n) + 10(R_{n-1} + R_n) \\ &= 11xS_n - 10S_n + (11+x) - 11x(R_1 + R_n) + 10(R_{n-1} + R_n)\ . \end{aligned}$$

So

$$11xS_n - 11S_n = 11x(R_1 + R_n) - 10(R_{n-1} + R_n) - (11+x)\ ,$$

which is equivalent to

$$S_n(11x - 11) = R_{n+1} - 10R_n + 10x - 11\ ,$$

and we get the result. □

In Equation 22, we can't do it $x = 1$, but we have $s_n = r_1 + r_2 + \cdots + r_n$, and it follows from the Proposition 2 that

$$s_n = \frac{10(10^n - 1) - 9n}{81} = \frac{10^{n+1} - 9n - 10}{81}\ ,$$

which represents the partial sum of n terms of $\{r_n\}_{n\geq 0}$.

It is worth mentioning that the series S_n does not converge to any value of x, which can be verified using the d'Alembert criterion (or ratio test), that is, determining the value

L of the limit (see [9]):

$$\lim_{n\to\infty}\left|\frac{R_{n+1}}{R_n}\right| = L. \tag{23}$$

Note that, considering α and β real numbers (positive or negative). That is, $x \in \left(-\infty, \frac{2\sqrt{10}}{11}\right) \cup \left(\frac{2\sqrt{10}}{11}, \infty\right)$. In the particular case where x is positive, that is, $x > \frac{2\sqrt{10}}{11}$, we have $L = |\alpha|$ and $L = |\beta|$ in the other case. In both cases, it is impossible for $L < 1$, which implies the divergence of the series. The case $L = 1$ also does not occur convergence.

5 Considerations

According [2] the polynomials are indispensable mathematical instruments, as they are straightforwardly defined and can be rapidly calculated on computer systems, they can be differentiated and integrated with ease. This reminder of the importance of polynomials and the fact that there is no polynomial associated with this particular Horadam-type sequence in the literature motivated our study. So, in this work, it was analyzed a particular case of a Horadam polynomial sequence, and our contribution to discrete mathematics is the specification from the results for a repunit polynomial sequence.

In general, the roots of polynomial equations of degree n become more difficult to find as n increases, and for $n \geq 5$, no general formula can be applied. Here, we present the repunit polynomials of degree n. However, it is not the focus of the work to determine the roots of these polynomials.

The results analyzed in this work aim to present some connections between a class of polynomials, the repunit polynomials, and the numerical sequence of repunit. They are highlighting the generating function of the repunit polynomial.

In the context of future work, we aim to carry out a thorough investigation into the matrix representation of this sequence, its generating functions, and a characterization of the roots of the repunit polynomial.

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