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# Modal laws in many-valued systems

## Leis modais em sistemas multivalorados

## Leyes modales en sistemas multivalentes

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**Abstract**. Since Dugundji, it has been known that there is no finite matrix semantics for modal logics between **S1** and **S5**. However, it remains interesting to know what can be valid among the modal laws relative to many-valued matrices. The logic **PM4N** was introduced by Jean-Yves Beziau as a modal and 4-valued system, planned to accept several modal laws. From that matrix semantics, the paper shows some valid results. In this paper, we compare the system **PM4N** with two well-known logics: the usual modal system **S5** and the paraconsistent logic $J_3$. We show that the set of **S5** theorems is properly included in the set of **PM4N** theorems; and every theorem of **PM4N** is a theorem of $J_3$.

**Keywords**. Modal laws, four valued logics, modal logics, paraconsistent logics.

**Resumo**. Desde Dugundji, é conhecido que não há semântica matricial finita para lógicas modais entre **S1** e **S5**. No entanto, ainda é interessante saber o que pode ser válido entre as leis modais em relação à matrizes multivaloradas. A lógica **PM4N** foi introduzida por Jean-Yves Beziau como um sistema modal de 4 valores, planejado para aceitar várias leis modais. A partir dessa semântica matricial, o artigo apresenta alguns resultados válidos. Neste artigo, comparamos o sistema **PM4N** com duas lógicas bem conhecidas: o sistema modal usual **S5** e a lógica paraconsistente $J_3$. Mostramos que o conjunto de teoremas de **S5** está propriamente incluso no conjunto de teoremas de **PM4N**; e todo teorema de **PM4N** é teorema de $J_3$.

**Palavras-chave**. Leis modais, lógicas tetra valoradas, lógicas modais, lógicas paraconsistentes.

**Resumen**. Desde Dugundji, se sabe que no existe una semántica matricial finita para lógicas modales entre **S1** y **S5**. Sin embargo, resulta interesante conocer qué puede ser válido entre las leyes modales en relación con matrices multivalentes. La lógica **PM4N** fue introducida por Jean-Yves Beziau como un sistema modal de 4 valores, diseñado para aceptar varias leyes modales. A partir de esta semántica matricial, el artículo presenta algunos resultados válidos. En este artículo, comparamos el sistema **PM4N** con dos lógicas bien conocidas: el sistema modal usual **S5** y la lógica paraconsistente $J_3$. Mostramos que el conjunto de teoremas de **S5** está adecuadamente incluido en el conjunto de teoremas de **PM4N**; y que todo teorema de **PM4N** es un teorema de $J_3$.

**Palabras-clave**. Leyes modales, lógicas de cuatro valores, lógicas modales, lógicas paraconsistentes.

**Mathematics Subject Classification (MSC):** 03B45.

# 1 Introduction

The logic **PM4N** was introduced by Beziau [3] as a basic 4-valued and modal logic. The system was proposed to keep many important modal notions.

In this paper, we present the system **PM4N** with the same matrix semantic, some simple variation on the symbols, and, as Beziau [3], we can show several modal laws valid in **PM4N**, but also laws and rules not valid in this 4-valued system. These validities and non-validities in **PM4N** are all verified in $\mathcal{T}_{\mathbf{PM4N}}$, the tableaux system for this logic presented in [15]. It is relevant for the comparison with **S5**.

Additionally, we eliminate one of the four values, exactly a non-designated element, to keep the true part of the system, and observe that we obtain the paraconsistent logic $J_3$. Thus, we link the two systems and investigate some modal aspects also in this 3-valued system.

In Section 2, we present the logic **PM4N**. In the next section, we show the validity of many modal laws and compare this logic with the modal logic **S5**. In the last section, by the exclusion of one value, a non-designated value, we obtain a paraconsistent 3-valued system, which coincides with the logic $J_3$ ([9] and [10]). Then we can compare the set of theorems of the two logics using some notions of translations between logics.

# 2 The logic PM4N

The logic **PM4N** is a propositional, modal and four-valued system. It is constituted, in a usual way, over a propositional language with the following set of operators $L = \{\vee, \neg, \Box\}$, in which the propositional operators $\vee$, $\neg$ and $\Box$ denote, respectively, the notions of disjunction, negation and necessity.

By simplicity, we use the same symbols from the propositional language also in the semantic structure for **PM4N**. Beziau [3] picked as basic also the operator $\wedge$, but considering that $\wedge$ and $\vee$ are inter-definable in **PM4N**, we have selected only the three above operators.

The original semantic for **PM4N** is defined by the following matrix semantics:

$$\mathcal{M}_{\mathbf{PM4N}} = (\{0, n, b, 1\}, \vee, \neg, \Box, \{b, 1\}).$$

Then, for $\mathcal{M}_{\mathbf{PM4N}}$, the elements $b$ and $1$ are the designated values and $0$ and $n$ are the non-designated values. As usual, we can indicate the set of designated or true values by $D = \{b, 1\}$.

Before showing the tables for these operators, let us stress the algebraic motivation, since we imagine that with these notions, the understanding of tables will be immediate. We must consider these four values disposed in a Boolean algebra of four elements as this following Hasse diagram.

$$\begin{array}{ccccc} & & 1 & & \\ & / & & \backslash & \\ n & & & & b \\ & \backslash & & / & \\ & & 0 & & \end{array}$$

The disjunction is interpreted as the supremum, that is, for $x, y \in \{0, n, b, 1\}$, the element $x \vee y = sup\{x, y\}$. The negation corresponds to the boolean complement, and the necessitation ensures that only the value $1$ is necessarily true.

The meanings of the basic operators are in the following tables.

| $\vee$ | 0 | n | b | 1 |
|---|---|---|---|---|
| 0 | 0 | n | b | 1 |
| n | n | n | 1 | 1 |
| b | b | 1 | b | 1 |
| 1 | 1 | 1 | 1 | 1 |

| | $\neg$ |
|---|---|
| 0 | 1 |
| n | b |
| b | n |
| 1 | 0 |

| | $\Box$ |
|---|---|
| 0 | 0 |
| n | 0 |
| b | 0 |
| 1 | 1 |

We have a four-valued logic, and as usual, the values $0$ and $1$ represent the *falsun* and the *verun*, but $b$ and $n$ are two complementary values that are not in a linear order. This negation is a type of classical negation, since it maps true values into false values and maps false values into true values.

This algebraic structure of **PM4N** is isomorphic to the Boolean algebra of four elements, $\mathcal{P}(2) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$, generated by a set of two elements $2 = \{0, 1\}$.

The formal matrix semantics of **PM4N** is the following.

$Var(\textbf{PM4N}) = \{p_1, p_2, p_3, ...\}$ is the set of the propositional variables of **PM4N** and $For(\textbf{PM4N})$ is the set of the formulas of **PM4N**, defined in the usual way.

**Definition 1.** *A restrict valuation for **PM4N** is a function*

$$v : Var(\textbf{\textit{PM4N}}) \to \{0, n, b, 1\}.$$

**Definition 2.** *A valuation for **PM4N** is a function that extends, in an unique way, the restrict valuation for the whole set For(**PM4N**), according to the above matrices.*

**Definition 3.** *A formula $\varphi \in For(\textbf{\textit{PM4N}})$ is valid in $\mathcal{M}_{PM4N}$ if, for every **PM4N**-valuation $v$, $v(\varphi) \in D$.*

**Definition 4.** *For $\Gamma \cup \{\varphi\} \subseteq For(\textbf{\textit{PM4N}})$, the set of formulas $\Gamma$ implies logically the formula $\varphi$, or $\varphi$ is a semantic consequence of $\Gamma$, if, for every **PM4N**-valuation $v$, always that $v(\Gamma) \subseteq D$, it follows that $v(\varphi) \in D$.*

Thus, we have that for every valuation $v$:

$$\Gamma \vDash \varphi \iff (v(\Gamma) \subseteq D \Rightarrow v(\varphi) \in D).$$

Besides these primitive and basic operators, we can define the following operators in **PM4N**.

Possibility: $\Diamond x =_{def} \neg\Box\neg x$

Conjunction: $x \wedge y =_{def} \neg(\neg x \vee \neg y)$

Conditional: $x \to y =_{def} \neg x \vee y$

Biconditional: $x \leftrightarrow y =_{def} (x \to y) \wedge (y \to x)$

Consistency: $\circ x =_{def} \Box x \vee \neg\Diamond x$

Inconsistency (or Contingency): $\bullet x =_{def} \Diamond x \wedge \neg\Box x$

Paraconsistent negation: $\sim x =_{def} \neg x \leftrightarrow \circ x$

The meanings of these new operators are given by the following tables.

| | $\Diamond$ |
|---|---|
| 0 | 0 |
| n | 1 |
| b | 1 |
| 1 | 1 |

| $\wedge$ | 0 | n | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| n | 0 | n | 0 | n |
| b | 0 | 0 | b | b |
| 1 | 0 | n | b | 1 |

| $\rightarrow$ | 0 | n | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| n | b | 1 | b | 1 |
| b | n | n | 1 | 1 |
| 1 | 0 | n | b | 1 |

| $\leftrightarrow$ | 0 | n | b | 1 |
|---|---|---|---|---|
| 0 | 1 | b | n | 0 |
| n | b | 1 | 0 | n |
| b | n | 0 | 1 | b |
| 1 | 0 | n | b | 1 |

| | $\circ$ |
|---|---|
| 0 | 1 |
| n | 0 |
| b | 0 |
| 1 | 1 |

| | $\bullet$ |
|---|---|
| 0 | 0 |
| n | 1 |
| b | 1 |
| 1 | 0 |

| | $\sim$ |
|---|---|
| 0 | 1 |
| n | n |
| b | b |
| 1 | 0 |

In modal logics, a possible but not necessary proposition is called contingent, and in the logic **PM4N**, its meaning coincides with the meaning of inconsistency of logics of formal inconsistency (LFI). The operators of consistency and inconsistency are complementary, and an interpretation for the consistency is that the classical values $\{0, 1\}$ are consistent, while the non-classical $\{n, b\}$ are not consistent.

Finally, the negation $\sim$ is paraconsistent because it maps the true value $b$ into $b$ and the false value $n$ into $n$. So in some cases, a proposition and its negation can both be true.

# 3 Comparing PM4N and S5

The logic **PM4N** was planned to keep a lot of modal laws. However, since Dugundji, 1940 ([11], [8]), we know that there is no finite matrix semantics for the hierarchy of modal logics between **S1** and **S5** and, after Dugundji and Kripke, between **K** and **S5**.

Now, considering the completely adequate tableaux system for **PM4N** presented in [15], we can verify the validity of several modal sentences, as well as the non-validity of other central modal results. We will compare the set of theorems of **PM4N** and **S5**.

Considering Mendelson [16] and Chellas [7], the logic **S5** can be determined by the following axiomatic system:

$(Ax_1)$ $\psi \rightarrow (\varphi \rightarrow \psi)$

$(Ax_2)$ $(\psi \rightarrow (\varphi \rightarrow \sigma)) \rightarrow ((\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \sigma))$

$(Ax_3)$ $(\neg\psi \rightarrow \varphi) \rightarrow ((\neg\psi \rightarrow \neg\varphi) \rightarrow \psi))$

(MP) $\psi, \psi \rightarrow \sigma$ / $\sigma$

The Boolean part plus:

(Definition) $\Diamond\psi \leftrightarrow \neg\Box\neg\psi$

(K) $\Box(\psi \rightarrow \varphi) \rightarrow (\Box\psi \rightarrow \Box\varphi)$

(T) $\Box\psi \to \psi$

(5) $\Diamond\psi \to \Box\Diamond\psi$

(Nec) $\vdash \psi$ / $\vdash \Box\psi$.

**Proposition 1.** *All the following modal formulas are valid in **PM4N**.*

| | |
|---|---|
| *(i) **K**:* $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ | *(ii) **D**:* $\Box\varphi \to \Diamond\varphi$ |
| *(iii) **T**:* $\Box\varphi \to \varphi$ | *(iv) **T'**:* $\varphi \to \Diamond\varphi$ |
| *(v) **4**:* $\Box\varphi \to \Box\Box\varphi$ | *(vi) **4'**:* $\Diamond\Diamond\varphi \to \Diamond\varphi$ |
| *(vii) **5**:* $\Diamond\varphi \to \Box\Diamond\varphi$ | *(viii) **5'**:* $\Diamond\Box\varphi \to \Box\varphi$ |
| *(ix) **B**:* $\varphi \to \Box\Diamond\varphi$ | *(x) **B'**:* $\Diamond\Box\varphi \to \varphi$ |
| *(xi) **M**:* $\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$ | *(xii) **C**:* $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ |
| *(xiii) **R**:* $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$ | *(xiv) **R'**:* $\Diamond\varphi \vee \Diamond\psi \leftrightarrow \Diamond(\varphi \vee \psi)$. |

*Proof.* For economy, we are going to omit the tableaux for these formulas. These validities can be found in ([15], p. 44-48). $\Box$

**Proposition 2.** *The following modal formulas and rules are not valid in **PM4N**.*

*(i)* $\nVdash \Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi$

*(ii)* $\nVdash \Diamond\varphi \wedge \Diamond\psi \to \Diamond(\varphi \wedge \psi)$

*(iii) **RN**:* $\varphi \nVdash \Box\varphi$*, even when* $\varphi$ *is valid in **PM4N**.*

*(iv) **RM**:* $\varphi \to \psi \nVdash \Box\varphi \to \Box\psi$

*(v) **RM'**:* $\varphi \to \psi \nVdash \Diamond\varphi \to \Diamond\psi$

*Proof.* (i) As a counter-example, take the values $v(\varphi) = b$ and $v(\psi) = n$. If $v(\varphi) = b$ and $v(\psi) = n$, then $v(\Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi) = \Box(b \vee n) \to \Box b \vee \Box n = \Box 1 \to 0 \vee 0 = 1 \to 0 = 0$. (ii) Like in (i), just take the values $v(\varphi) = b$ and $v(\psi) = n$ and so $v(\Diamond\varphi \wedge \Diamond\psi \to \Diamond(\varphi \wedge \psi)) = 0$. (iii) A non-closed tableau for this formula can be found in [15]. (iv) For $v(\varphi) = 1$ and $v(\psi) = b$, we have $1 \to b$ and $1 \to 0$, then $v(\varphi \to \psi) = b$ and $v(\Box\varphi \to \Box\psi) = 0$. (v) For $v(\varphi) = n$ and $v(\psi) = 0$, we have $n \to 0$ and $1 \to 0$, then $v(\varphi \to \psi) = b$ and $v(\Diamond\varphi \to \Diamond\psi) = 0$. $\Box$

The non-validity of these rules makes the conception of the logic **PM4N** a little strange. A valid conditional does not transfer its validity to these several cases.

At first, we had imagined that **PM4N** would have fewer theses than **S5**, exactly because the two systems share the Boolean part, every modal usual axiom of **S5** is valid in the $\mathcal{M}_{\textbf{PM4N}}$, and some of the usual rules for **S5** are not valid in **PM4N**. But it is not the case, and as Beziau [3] has mentioned, all of the theorems of **S5** are theorems of **PM4N**.

Furthermore, the formula $\beta$:

$$\Box\varphi \vee \Box(\varphi \to \psi) \vee \Box(\varphi \to \neg\psi)$$

is not a theorem of **S5**, but it is valid in **PM4N**.

The proof that this formula is not a **S5**-theorem is in [18] and below we can see that it is valid in **PM4N**.

| $\varphi$ | $\psi$ | $\varphi \to \psi$ | $\varphi \to \neg\psi$ | $\Box\varphi$ | $\Box(\varphi \to \psi)$ | $\Box(\varphi \to \neg\psi)$ | $\beta$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 0 | n | 1 | 1 | 0 | 1 | 1 | 1 |
| 0 | b | 1 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| n | 0 | b | 1 | 0 | 0 | 1 | 1 |
| n | n | 1 | b | 0 | 1 | 0 | 1 |
| n | b | b | 1 | 0 | 0 | 1 | 1 |
| n | 1 | 1 | b | 0 | 1 | 0 | 1 |
| b | 0 | n | 1 | 0 | 0 | 1 | 1 |
| b | n | n | 1 | 0 | 0 | 1 | 1 |
| b | b | 1 | n | 0 | 1 | 0 | 1 |
| b | 1 | 1 | n | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | n | n | b | 1 | 0 | 0 | 1 |
| 1 | b | b | n | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |

**Definition 5.** *A valid formula in which its last column contains only the value* $1$ *is called necessarily true.*

The formula $\beta$ is necessarily true.

So, we will see that the set of **S5**-theorems is included in the set of **PM4N**-theorems (or valid formulas of **PM4N**).

All schemes of axioms of **S5** are formulas necessarily true, as we can see in some cases. The classical axioms are classically valid in any Boolean algebra, and the model of 4-values $(\{0, n, b, 1\}, \vee, \neg, \{0, 1\})$ is a Boolean algebra. The MP takes tautologies into tautologies.

| $\psi$ | $\Box\psi$ | $\Diamond\psi$ | **Def** | **K** | **5** |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 |
| n | 0 | 1 | 1 | 1 | 1 |
| n | 0 | 1 | 1 | 1 | 1 |
| n | 0 | 1 | 1 | 1 | 1 |
| n | 0 | 1 | 1 | 1 | 1 |
| b | 0 | 1 | 1 | 1 | 1 |
| b | 0 | 1 | 1 | 1 | 1 |
| b | 0 | 1 | 1 | 1 | 1 |
| b | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

**Theorem 1.** *Every theorem of **S5** is valid in **PM4N**.*

*Proof.* Each axiom above is necessarily true, and the rules (MP) and (Nec) take necessarily true sentences into necessarily true sentences. Thus, all of the theorems of **S5** are necessarily true. □

Finally, we have that every **S5**-theorem is valid in the matrix semantics for **PM4N**, and **PM4N** has valid formulas that are not valid for **S5**. Thus, we have a proper inclusion in the set of theorems.

## 4 The relation between PM4N and J3

When we look for a four-valued logic similar to **PM4N**, with two distinguished elements $D = \{b, 1\}$, we can think that we have a rank for the true sentences, the necessarily true (only the value $1$) and the possibly true (values $b$ and $1$). But the group of non-true sentences is a complementary set.

It seems that if we collapse the elements $0$ and $n$, the valid sentences remain almost the same. Thus, we will develop a reflection with a three-valued semantic disposed in a partial order without the element $n$.

So we start from:

$$\mathcal{M}_{\mathbf{LM3N}} = (\{0, b, 1\}, \vee, \neg, \Box, \{b, 1\}),$$

such that $D = \{b, 1\}$ and consider the order:

$$\begin{array}{ccc} 1 & & \\ | & \diagdown & \\ | & & b \\ | & \diagup & \\ 0 & & \end{array} \qquad \looparrowright \qquad \begin{array}{c} 1 \\ | \\ b \\ | \\ 0 \end{array}$$

A first observation is that with exactly three elements, the partial order loses relevance, because any two elements are comparable and thus the order is linear.

This way, we obtain a three-valued paraconsistent logic, following the original intuitions, in the propositional language $L = \{\neg, \Box, \vee\}$, in which the first two operators are unary and the last one is binary, with the following tables.

| | $\neg$ |
|---|---|
| 0 | 1 |
| b | 0 |
| 1 | 0 |

| $\vee$ | 0 | b | 1 |
|---|---|---|---|
| 0 | 0 | b | 1 |
| b | b | b | 1 |
| 1 | 1 | 1 | 1 |

| | $\Box$ |
|---|---|
| 0 | 0 |
| b | 0 |
| 1 | 1 |

With the three elements $\{0, b, 1\}$, we don't have a Boolean algebra anymore, but considering that the negation $\neg$ of **PM4N** is classical, then we take $\neg b = 0$.

Using similar definitions for other operators, we have the following ones with the respective tables:

Possibility: $\Diamond x =_{def} \neg\Box\neg x$

Conjunction: $x \wedge y =_{def} inf\{x, y\}$

Conditional: $x \to y =_{def} \neg x \vee y$

Biconditional: $x \leftrightarrow y =_{def} (x \to y) \wedge (y \to x)$

Consistency: $\circ x =_{def} \Box x \vee \neg\Diamond x$

Inconsistency (or Contingency): $\bullet x =_{def} \Diamond x \wedge \neg\Box x$

Paraconsistent negation: $\sim x =_{def} x \leftrightarrow \bullet x$.

| | $\Diamond$ |
|---|---|
| 0 | 0 |
| b | 1 |
| 1 | 1 |

| $\wedge$ | 0 | b | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| b | 0 | b | b |
| 1 | 0 | b | 1 |

| $\to$ | 0 | b | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| b | 0 | b | 1 |
| 1 | 0 | b | 1 |

| $\leftrightarrow$ | 0 | b | 1 |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| b | 0 | b | b |
| 1 | 0 | b | 1 |

| | $\circ$ |
|---|---|
| 0 | 1 |
| b | 0 |
| 1 | 1 |

| | $\bullet$ |
|---|---|
| 0 | 0 |
| b | 1 |
| 1 | 0 |

| | $\sim$ |
|---|---|
| 0 | 1 |
| b | b |
| 1 | 0 |

If we take as primitive the operators $\sim$, $\Diamond$ and $\vee$, we can define all these other operators as:

Necessary: $\Box x =_{def} \sim \Diamond \sim x$

Classical negation: $\neg x =_{def} \sim \Diamond x$,

and observe that this 3-valued restriction generates exactly the well-known paraconsistent logic $J_3$.

The logic $J_3$ was introduced by D'Ottaviano and da Costa [9], in 1970, from this three-valued matrix semantics, involving aspects of the recently created paraconsistent logics. The system $J_3$ has been studied by other authors as [1], [2], [13] and [19] and with different motivations, denominations and approaches [4] and [6].

Carnielli and Marcos [5] and Carnielli, Coniglio and Marcos [4] introduced a family of paraconsistent logics, named the logics of formal inconsistency (LFI's), in which the consistency operator occurs in the language as a primitive operator; and the first system of the family of LFI's, the LFI1 logic, coincides with the logic $J_3$. The consistency operator or the operator of 'good behaviour' points that if the proposition assumes only the values $0$ or $1$, then it is well-behaved or acts accordingly to the classical logic.

The three values of $J_3$, given by the set $\{0, \frac{1}{2}, 1\}$, were thought in a linear order, however for only three elements there are no variances between the partial and linear order.

Considering that we are interested in obtaining deductive systems for **PM4N**, we will consider the following axiomatic system for $J_3$ [13].

Schemes of Axioms:

(A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$

(A2) $(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma))$

(A3) $(\varphi \wedge \psi) \rightarrow \varphi$

(A4) $(\varphi \wedge \psi) \rightarrow \psi$

(A5) $(\sigma \rightarrow \varphi) \rightarrow ((\sigma \rightarrow \psi) \rightarrow (\sigma \rightarrow (\varphi \wedge \psi)))$

(A6) $\varphi \rightarrow (\varphi \vee \psi)$

(A7) $\psi \rightarrow (\varphi \vee \psi)$

(A8) $(\varphi \rightarrow \sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow ((\varphi \vee \psi) \rightarrow \sigma))$

(A9) $\sim\sim \varphi \leftrightarrow \varphi$

(A10) $\varphi \vee (\varphi \to \psi)$
(A11) $\circ\varphi \to (\varphi \to (\sim \varphi \to \psi))$
(A12) $\sim \circ\varphi \to (\varphi \wedge \sim \varphi)$
(A13) $\circ\varphi \to \circ \sim \varphi$
(A14) $(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \to \psi)$
(A15) $(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \vee \psi)$.

Deduction rules:
(MP) $\varphi, \varphi \to \psi \vdash \psi$.

The axioms (A1) - (A8) plus the rule MP point that $J_3$ is constructed over the positive logic. The axioms (A9) and (A10) formalize aspects of negation of $J_3$. The axioms (A11) - (A15) make explicit the paraconsistent aspects of $J_3$.

The validity of the $J_3$-propositions can be tested through the three-valued matrix, or with the following 3-valued tableaux system adapted from **PM4N** [15].

**A 3-valued tableaux system for $J_3$ or LM3N**

$t$ expansion:

$$\frac{t \quad \varphi}{b\ \varphi \quad | \quad 1\ \varphi}$$

Negation:

$$\frac{1 \qquad \neg\varphi}{0 \qquad \varphi} \qquad\qquad \frac{0 \qquad \neg\varphi}{t \qquad \varphi}$$

Paraconsistent negation:

$$\frac{0 \qquad \sim\varphi}{1 \qquad \varphi} \qquad\qquad \frac{b \qquad \sim\varphi}{b \qquad \varphi} \qquad\qquad \frac{1 \qquad \sim\varphi}{0 \qquad \varphi}$$

Conjunction:

$$\frac{0 \qquad \varphi\wedge\psi}{0\ \varphi \quad | \quad 0\ \psi} \qquad\qquad \frac{b \qquad \varphi\wedge\psi}{\begin{array}{ccc} b\ \varphi & | & 1\ \varphi \\ t\ \psi & | & b\ \psi \end{array}} \qquad\qquad \frac{1 \quad \varphi\wedge\psi}{\begin{array}{cc} 1 & \varphi \\ 1 & \psi \end{array}}$$

Disjunction:

$$\begin{array}{ll} 0 & \varphi \vee \psi \\ \hline 0 & \varphi \\ 0 & \psi \end{array} \qquad \begin{array}{ccc} b & \varphi \vee \psi & \\ \hline \multicolumn{2}{c|}{b\ \varphi} & 0\ \varphi \\ 0\ \psi & \multicolumn{1}{|c|}{b\ \psi} & b\ \psi \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{1 \quad \varphi \vee \psi} \\ \hline 1\ \varphi & 1\ \psi \end{array}$$

Conditional:

$$\begin{array}{ll} 0 & \varphi \to \psi \\ \hline 0 & \psi \\ t & \varphi \end{array} \qquad \begin{array}{ll} b & \varphi \to \psi \\ \hline b & \psi \\ t & \varphi \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{1 \quad \varphi \to \psi} \\ \hline 0\ \varphi & 1\ \psi \end{array}$$

Modal operators:

$$\begin{array}{c|c} \multicolumn{2}{c}{0 \quad \Box\varphi} \\ \hline 0\ \varphi & b\ \varphi \end{array} \qquad \begin{array}{ll} 1 & \Box\varphi \\ \hline 1 & \varphi \end{array} \qquad \begin{array}{ll} 0 & \Diamond\varphi \\ \hline 0 & \varphi \end{array} \qquad \begin{array}{ll} 1 & \Diamond\varphi \\ \hline t & \varphi \end{array}$$

Consistency operator:

$$\begin{array}{ll} 0 & \circ\varphi \\ \hline b & \varphi \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{1 \quad \circ\varphi} \\ \hline 0\ \varphi & 1\ \varphi \end{array}$$

**Definition 6.** *A branch of a tableau of $\mathcal{T}_{LM3N}$ or $\mathcal{T}_{J3}$ is closed if the marked formulas occur in the path:*

(i) $k_1\ \ \varphi$ *and* $k_2\ \ \varphi$*, for any formula* $\varphi$ *and* $k_1 \neq k_2$*;*

(ii) $b\ \ *\varphi$*, for any formula* $*\varphi$*, with* $* \in \{\Box, \Diamond, \circ\}$.

**Definition 7.** *A tableau of $\mathcal{T}_{LM3N}$ or $\mathcal{T}_{J3}$ is closed if all of its branches are closed.*

(A9) $\Vdash\ \sim\sim \varphi \leftrightarrow \varphi$

(i) $\Vdash\ \sim\sim \varphi \to \varphi$

$$\begin{array}{cc} \multicolumn{2}{c}{0 \quad \sim\sim \varphi \to \varphi} \\ \multicolumn{2}{c}{0 \quad \varphi} \\ \multicolumn{2}{c}{t \quad \sim\sim \varphi} \\ \swarrow & \searrow \\ b \quad \sim\sim \varphi & 1 \quad \sim\sim \varphi \\ b \quad \sim \varphi & 0 \quad \sim \varphi \\ b \quad \varphi & 1 \quad \varphi \\ \times & \times \end{array}$$

(ii) $\Vdash\ \varphi \to\sim\sim \varphi$

$$\begin{array}{c} 0 \quad \varphi \to\sim\sim \varphi \\ 0 \quad \sim\sim \varphi \\ t \quad \varphi \\ 1 \quad \sim \varphi \\ 0 \quad \varphi \\ \times \end{array}$$

(A10) $\Vdash \ \varphi \vee (\varphi \to \psi)$

$$\begin{array}{c}
0 \quad \varphi \vee (\varphi \to \psi) \\
0 \quad \varphi \\
0 \quad \varphi \to \psi \\
0 \quad \psi \\
t \quad \varphi \\
\times
\end{array}$$

(A13) $\Vdash \ \circ\varphi \to \circ \sim \varphi$

$$\begin{array}{c}
0 \quad \circ\,\varphi \to \circ \sim \varphi \\
0 \quad \circ \sim \varphi \\
t \quad \circ\,\varphi \\
b \quad \sim \varphi \\
b \quad \varphi \\
\begin{array}{cc}
\begin{array}{c} b \quad \circ\,\varphi \\ \times \end{array} &
\begin{array}{c} 1 \quad \circ\,\varphi \\
\begin{array}{cc}
\begin{array}{c} 0 \quad \varphi \\ \times \end{array} &
\begin{array}{c} 1 \quad \varphi \\ \times \end{array}
\end{array}
\end{array}
\end{array}
\end{array}$$

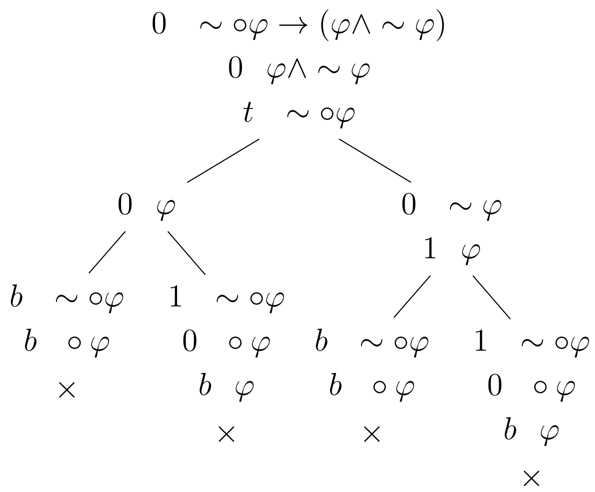
(A11) $\Vdash \ \circ\varphi \to (\varphi \to (\sim \varphi \to \psi))$

$$\begin{array}{c}
0 \quad \circ\,\varphi \to (\varphi \to (\sim \varphi \to \psi)) \\
0 \quad \varphi \to (\sim \varphi \to \psi) \\
t \quad \circ\,\varphi \\
0 \quad \sim \varphi \to \psi \\
t \quad \varphi \\
0 \quad \psi \\
t \quad \sim \varphi \\
\begin{array}{cc}
\begin{array}{c} b \quad \circ\,\varphi \\ \times \end{array} &
\begin{array}{c} 1 \quad \circ\,\varphi \\
\begin{array}{cc}
\begin{array}{c} 0 \quad \varphi \\ \times \end{array} &
\begin{array}{c} 1 \quad \varphi \\
\begin{array}{cc}
\begin{array}{c} b \quad \varphi \\ \times \end{array} &
\begin{array}{c} 1 \quad \varphi \\
\begin{array}{cc}
\begin{array}{c} b \quad \sim \varphi \\ b \quad \varphi \\ \times \end{array} &
\begin{array}{c} 1 \quad \sim \varphi \\ 0 \quad \varphi \\ \times \end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}$$

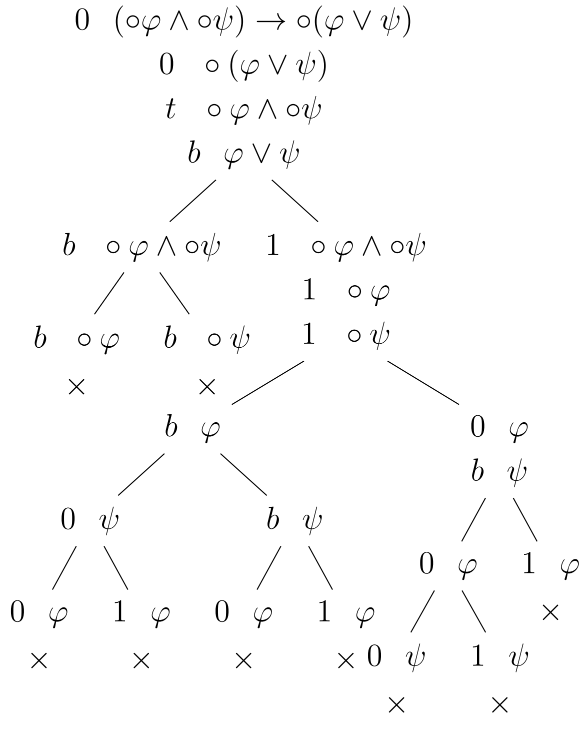
(A14) $\Vdash \ (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \to \psi)$

$$\begin{array}{c}
0 \quad (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \to \psi) \\
0 \quad \circ\,(\varphi \to \psi) \\
t \quad \circ\,\varphi \wedge \circ\psi \\
b \quad \varphi \to \psi \\
b \quad \psi \\
t \quad \varphi \\
\begin{array}{cc}
\begin{array}{c} b \quad \circ\,\varphi \wedge \circ\psi \\
\begin{array}{cc}
\begin{array}{c} b \quad \circ\,\varphi \\ \times \end{array} &
\begin{array}{c} 1 \quad \circ\,\varphi \\ b \quad \circ\,\psi \\ \times \end{array}
\end{array}
\end{array} &
\begin{array}{c} 1 \quad \circ\,\varphi \wedge \circ\psi \\ 1 \quad \circ\,\varphi \\ 1 \quad \circ\,\psi \\
\begin{array}{cc}
\begin{array}{c} 0 \quad \psi \\ \times \end{array} &
\begin{array}{c} 1 \quad \psi \\ \times \end{array}
\end{array}
\end{array}
\end{array}
\end{array}$$

(A12) $\Vdash \sim \circ\varphi \rightarrow (\varphi \wedge \sim \varphi)$



(A15) $\Vdash \ (\circ\varphi \wedge \circ\psi) \rightarrow \circ(\varphi \vee \psi)$



## 4.1 On the axioms of $J_3$ into PM4N

Now, we adapt $\mathcal{T}_{\mathbf{PM4N}}$ presented in ([15], p. 42-44), the tableaux system of **PM4N**, for the operator of consistency and the paraconsistent negation. Of course, the new tableaux rules comes from the analysis of $\circ$-matrix and $\sim$-matrix given in the second section.

Consistency operator:

$$\frac{0 \quad \circ\varphi}{n\ \varphi \quad | \quad b\ \varphi} \qquad\qquad \frac{1 \quad \circ\varphi}{0\ \varphi \quad | \quad 1\ \varphi}$$

Paraconsistent negation:

$$\frac{0 \quad \sim\varphi}{1 \quad \varphi} \qquad\qquad \frac{n \quad \sim\varphi}{n \quad \varphi} \qquad\qquad \frac{b \quad \sim\varphi}{b \quad \varphi} \qquad\qquad \frac{1 \quad \sim\varphi}{0 \quad \varphi}$$

We need to include a specific condition to the closure of the new tableaux.

**Definition 8.** *A branch of a tableau of $\mathcal{T}_{PM4N}$ is closed if the marked formulas occur in the path:*

(i) $k_1\ \ \varphi$ *and* $k_2\ \ \varphi$*, for any formula* $\varphi$ *and* $k_1 \neq k_2$*;*

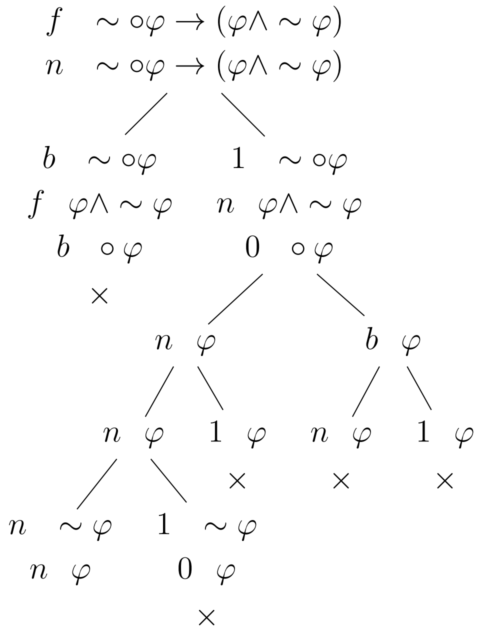
(ii) $n$ *or* $b\ \ *\varphi$*, for any formula* $*\varphi$*, with* $* \in \{\Box, \Diamond, \circ\}$.

(A12) $\nVdash\ \sim\circ\varphi \to (\varphi\wedge\sim\varphi)$

$$\begin{array}{c}
f \quad \sim\circ\varphi \to (\varphi\wedge\sim\varphi) \\
0 \quad \sim\circ\varphi \to (\varphi\wedge\sim\varphi) \\
1 \quad \sim\circ\varphi \\
0 \quad \varphi\wedge\sim\varphi \\
0 \quad \circ\varphi
\end{array}$$

$$\begin{array}{cc}
n \quad \varphi & b \quad \varphi
\end{array}$$

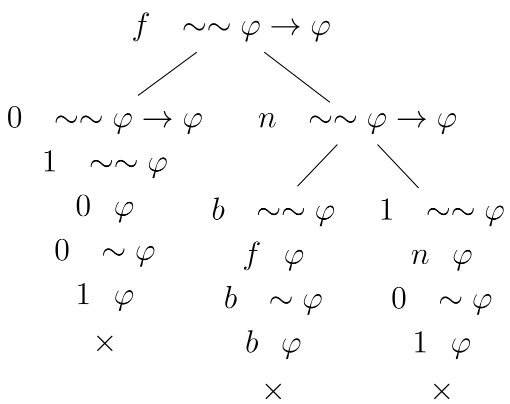
$$\begin{array}{cccc|cccc}
0\ \varphi & 0\ \sim\varphi & b\ \varphi & n\ \varphi & 0\ \varphi & 0\ \sim\varphi & b\ \varphi & n\ \varphi \\
\times & 1\ \varphi & n\ \sim\varphi & b\ \sim\varphi & \times & 1\ \varphi & n\ \sim\varphi & b\ \sim\varphi \\
 & \times & n\ \varphi & b\ \varphi & & \times & n\ \varphi & b\ \varphi \\
 & & \times & \times & & & \times & \times
\end{array}$$



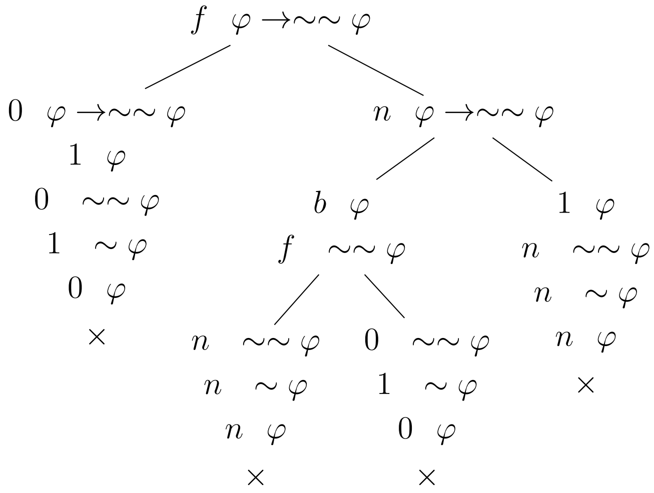
There is an open branch, such that $v(\varphi) = n$. So A12 does not hold.

(A9) $\Vdash \sim\sim \varphi \leftrightarrow \varphi$

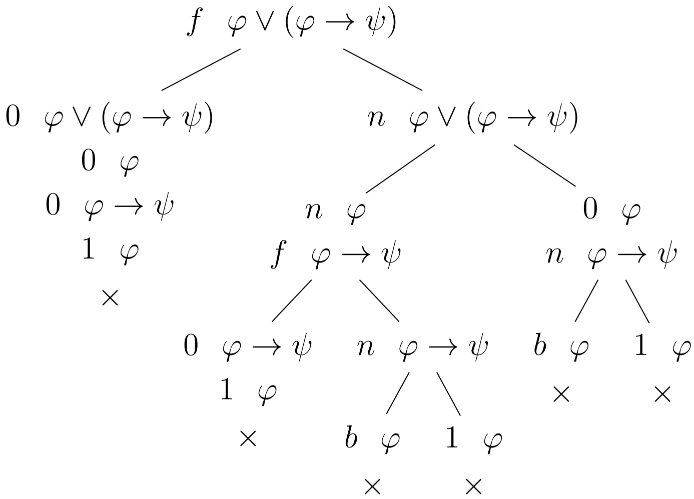
(i) $\Vdash \sim\sim \varphi \rightarrow \varphi$



(ii) $\Vdash \ \varphi \rightarrow \sim\sim \varphi$



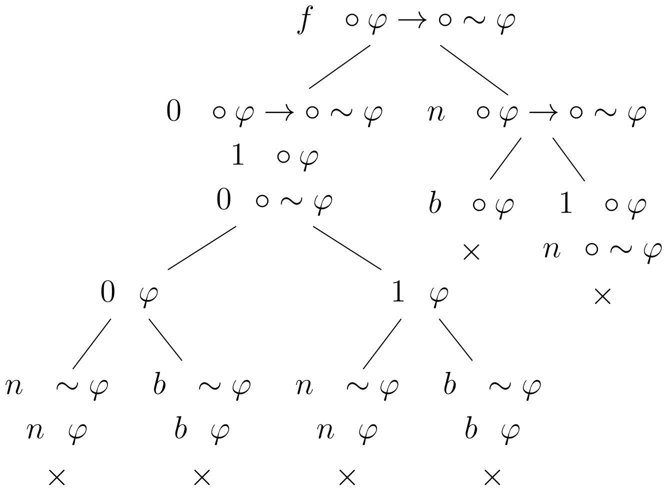

(A10) $\Vdash \varphi \vee (\varphi \rightarrow \psi)$




(A11) $\Vdash \circ\varphi \rightarrow (\varphi \rightarrow (\sim \varphi \rightarrow \psi))$

The law above is valid in **PM4N**.

(A13) $\Vdash \circ\varphi \rightarrow \circ \sim \varphi$




The next two laws are also valid in **PM4N**.
(A14) $\Vdash (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \to \psi)$
(A15) $\Vdash (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \vee \psi)$.

## 4.2 Comparing $J_3$ and PM4N

Using some notions about translations between logics, we show that every valid formula of **PM4N** is also valid in $J_3$, but $J_3$ has more theses than **PM4N**.

Da Silva, D'Ottaviano and Sette [17], in 1999, initiate the development of a theory of translations between logics. Their definition of translation between logics uses a very general characterization of logic.

**Definition 9.** *A consequence operator on a set $L$ is a function $C : \mathcal{P}(L) \to \mathcal{P}(L)$ such that, for every $X, Y \subseteq L$:*
(i) $X \subseteq C(X)$*;*
(ii) $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$*;*
(iii) $C(C(X)) \subseteq C(X)$.

This is the Tarski's consequence operator or closure operator.

**Definition 10.** *Abstract logic is a pair $\mathbb{L} = \langle L, C \rangle$, such that $L$ is any set, the domain of $\mathbb{L}$, and $C$ is a consequence operator on $L$.*

The following general definition of translation between logics was proposed by da Silva, D'Ottaviano and Sette [17].

**Definition 11.** *A translation from a logic $\mathbb{L}_1 = \langle L_1, C_1 \rangle$ into a logic $\mathbb{L}_2 = \langle L_2, C_2 \rangle$ is a function $t : L_1 \to L_2$ such that, for any $X \subseteq L_1$:*

$$t(C_1(X)) \subseteq C_2(t(X)).$$

The definition of abstract logic requests only set theory components. But in general we handle logics as pairs $\mathcal{L} = \langle L, C \rangle$, such that $L$ is a formal language and $C$ is a standard consequence operator in the set of formulas of $L$ denoted by $For(L)$.

**Definition 12.** *A logic system defined over L is a pair $\mathcal{L} = \langle L, C \rangle$, in which L is a formal language and $C$ is a standard consequence operator in the free algebra $For(L)$ of the formulas of L.*

When $\mathcal{L}_1$ and $\mathcal{L}_2$ are logic systems with associated syntactic consequence relations $\vdash_1$ and $\vdash_2$, respectively, the definition of translation between logics can be presented in terms of the consequence relations: $t$ is a translation from $\mathcal{L}_1$ into $\mathcal{L}_2$ if, and only if, for $\Gamma \cup \{\varphi\} \subseteq For(\mathcal{L}_1)$, it follows that:

$$\Gamma \vdash_1 \varphi \Rightarrow t(\Gamma) \vdash_2 t(\varphi).$$

Now, the conservative translations, that characterize a subclass of translations, introduced and investigated by Feitosa and D'Ottaviano (see [12] and [14]).

**Definition 13.** *For two logics $\mathbb{L}_1$ and $\mathbb{L}_2$, a conservative translation from $\mathbb{L}_1$ into $\mathbb{L}_2$ is a function $t : L_1 \to L_2$ such that, for every set $X \cup \{x\} \subseteq L_1$:*

$$x \in C_1(X) \Leftrightarrow t(x) \in C_2(t(X)).$$

**Definition 14.** *A conservative mapping from the logic $\mathbb{L}_1$ into the logic $\mathbb{L}_2$ is a function $t : \mathbb{L}_1 \to \mathbb{L}_2$ such that, for every $x \in L_1$:*

$$x \in C_1(\emptyset) \Leftrightarrow t(x) \in C_2(\emptyset).$$

In terms of the logical systems, given $\mathcal{L}_1$ and $\mathcal{L}_2$, a conservative translation is a function $t : For(\mathcal{L}_1) \to For(\mathcal{L}_2)$ such that, for every subset $\Gamma \cup \{\varphi\} \subseteq For(\mathcal{L}_1)$:

$$\Gamma \vdash_1 \varphi \Leftrightarrow t(\Gamma) \vdash_2 t(\varphi).$$

In order to show that every valid formula of **PM4N** is also valid in $J_3$, we will consider $\mathcal{L}_1 =$ **PM4N**, $\mathcal{L}_2 = J_3$ and $t$ as the identity function $\iota$.

**Proposition 3.** *The function $\iota$ is a translation from **PM4N** into $J_3$.*

*Proof.* We will show that for every $\Gamma \cup \{\varphi\} \subseteq For($**PM4N**$)$, if $\iota(\Gamma) \nvDash \iota(\varphi)$, then $\Gamma \nvDash \varphi$.

We take $J_3$ in the original language $L = \{\sim, \Diamond, \vee\}$ (it is not central).

For any formula of **PM4N**, it can be written in the same language with the definitions of new operators from Section 2.

As the function $\iota$ is the identity, then exactly the same formula will be tested in $J_3$.

If $\iota(\Gamma) \nvDash \iota(\varphi)$, then $\Gamma \nvDash \varphi$ and there is a $J_3$-valuation $e$ such that $e(\Gamma) \subseteq D$ and $e(\varphi) = 0$.

As we have used only the values $\{0, b, 1\}$ and the operations of $\Gamma \cup \{\varphi\}$ share exactly the same values, then we have that $\Gamma \nvDash \varphi$ in **PM4N**. $\square$

So we have that every valid formula of **PM4N** is also valid in $J_3$, and the axiom (A12) of $J_3$ is not valid in **PM4N**. This way there are more theses in the three-valued logic.

# 5 Final considerations

The aim of this paper was to understand and correlate the modal and 4-valued logic **PM4N** with other modal logics. As we were interested in the modal laws, we compared **PM4N** with the modal system **S5**. Additionally, we related the system **PM4N** with the paraconsistent logic $J_3$.

We also exposed the inclusion of theorems of **S5** into the class of valid formulas of **PM4N**. Futhermore, we showed that every theorem of **PM4N** is also a theorem of the paraconsistent logic $J_3$.

In a next article, we will try to relate the modal operators of **PM4N** as a pair of Galois.

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