Brazilian Electronic Journal of Mathematics, v.5, jan/dez, 2024. ISSN: 2675-1313

# A note on the Tetrarrin sequence

## Uma nota sobre a sequência de Tetrarrin

## Una nota sobre la secuencia de Tetrarrin

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**Resumo**. Este estudo explora uma extensão da sequência de Perrin, definindo a sequência de Tetrarrin. Assim, investigamos os números Tetrarrin e suas relações matemáticas, permitindo um aprofundamento na compreensão dos números de Perrin. Neste contexto, analisamos a fórmula de Binet, a forma matricial, entre outros teoremas, possibilitando a obtenção dos termos dessa nova sequência por diferentes métodos. É importante destacar que a sequência de Tetrarrin é de ordem superior e derivada da sequência de Perrin. Para trabalhos futuros, pretendemos integrar esse conteúdo com outras áreas de estudo, além de utilizar ferramentas e softwares que possibilitem a visualização das propriedades matemáticas dessa sequência de maneira mais intuitiva e acessível.

**Palavras-chave**. Extensão, sequência de Perrin, sequência de Tetrarrin.

**Abstract**. This study explores an extension of the Perrin sequence, defining the Tetrarrin sequence. Thus, we investigated Tetrarrin numbers and their mathematical relationships, allowing a deeper understanding of Perrin numbers. In this context, we analyze Binet?s formula, the matrix form, among other theorems, making it possible to obtain the terms of this new sequence using different methods. It is important to highlight that the Tetrarrin sequence is of higher order and derived from the Perrin sequence. For future work, we intend to integrate this content with other areas of study, in addition to using tools and software that enable the visualization of the mathematical properties of this sequence in a more intuitive and accessible way.

**Keywords**. Extension, Perrin sequence, Tetrarrin sequence.

**Resumen**. Este estudio explora una extensión de la secuencia de Perrin, definiendo la secuencia de Tetrarrin. Así, investigamos los números de Tetrarrin y sus relaciones matemáticas, lo que permitió una comprensión más profunda de los números de Perrin. En este contexto, analizamos la fórmula de Binet, la forma matricial, entre otros teoremas, permitiendo obtener los términos de esta nueva secuencia mediante diferentes métodos. Es importante resaltar que la secuencia Tetrarrin es de orden superior y deriva de la secuencia Perrin. Para futuros trabajos pretendemos integrar este contenido con otras áreas de estudio, además de utilizar herramientas y software que permitan visualizar las propiedades matemáticas de esta secuencia de una manera más intuitiva y accesible.

**Palabras-clave**. Extensión, secuencia de Perrin, secuencia de Tetrarrin.

**Mathematics Subject Classification (MSC):** 11B37, 11B39.

# 1 Introduction

Recent works are glimpsing the extension content of recurrent numerical sequences. We observe the case of Tribonacci sequence, Tetranacci and etc derived from the Fibonacci sequence [2, 5, 6].

Similarly, we have work on the Tridovan sequence, derived from the Padovan sequence [4]. These studies are recorded in the online encyclopedia [3].

Based on this, this research deals with the extension of a sequence similar to the Padovan sequence, called the Perrin sequence. The extension of the Perrin numbers is primarily called in this work the Tetrarrin sequence, which is fourth order and derived from the Perrin numbers.

It is noteworthy that the Perrin sequence has a recurrence relation given by: $R_n = R_{n-2} + R_{n-3}, n \geq 3$, with initial values $R_0 = 3, R_1 = 0, R_2 = 2$. It is then observed the insertion of one more term in the recurrence and in the initial value, attributing the study to the Tetrarrin numbers [1].

In the following section, this Tetrarrin sequence will be defined, as well as the mathematical relationships arising from its respective recurrence formula.

## 2 Tetrarrin sequence

Primarily defined in this research, the Tetrarrin sequence is a linear and recurrent sequence of the fourth order, being an order extension in relation to the Perrin sequence.

**Definition 1.** *The Tetrarrin sequence, represented by $R_{4(n)}$ with $n \geqslant 0$ and $n \in \mathbb{N}$, has the following recurrence formula:*

$$R_{4(n)} = R_{4(n-2)} + R_{4(n-3)} + R_{4(n-4)},$$

*with the following initial values: $R_{4(0)} = 3$, $R_{4(1)} = 0$, $R_{4(2)} = 2$ and $R_{4(3)} = 3$.*

Thus, the first terms of this sequence are: $3, 0, 2, 3, 5, 5, 10, 13, \ldots$

**Theorem 1.** *The matrix form of the Tetrarrin sequence, represented by $\Gamma^{(n)}$ for $n \geqslant 1$ and $n \in \mathbb{N}$, is expressed by:*

$$\boldsymbol{v_t}\Gamma^{(n)} = \begin{bmatrix} 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} R_{4(n)} & R_{4(n+1)} & R_{4(n+2)} & R_{4(n+3)} \end{bmatrix}.$$

*Where $\boldsymbol{v_t}$ is the row vector containing the initial values of the Tetrarrin sequence.*

*Proof.* Proving that for $n = 1$, the equality is true, we have:

$$\begin{aligned} \boldsymbol{v_t}\Gamma^{(1)} &= \begin{bmatrix} 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} R_{4(1)} & R_{4(2)} & R_{4(3)} & R_{4(4)} \end{bmatrix}. \end{aligned}$$

Assuming it holds for any $n = k, k \in \mathbb{N}$:

$$\boldsymbol{v_t}\Gamma^{(k)} = \begin{bmatrix} 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^k$$
$$= \begin{bmatrix} R_{4(k)} & R_{4(k+1)} & R_{4(k+2)} & R_{4(k+3)} \end{bmatrix}.$$

Then, verifying that it also holds for $n = k + 1$:

$$\begin{aligned} \boldsymbol{v_t}\Gamma^{(k+1)} &= \boldsymbol{v_t}\Gamma^{(k)}\Gamma \\ &= \begin{bmatrix} 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^k \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R_{4(k)} & R_{4(k+1)} & R_{4(k+2)} & R_{4(k+3)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R_{4(k+1)} & R_{4(k+2)} & R_{4(k+3)} & R_{4(k+2)} + R_{4(k+1)} + R_{4(k)} \end{bmatrix} \\ &= \begin{bmatrix} R_{4(k+1)} & R_{4(k+2)} & R_{4(k+3)} & R_{4(k+4)} \end{bmatrix}. \end{aligned}$$

Thus validating the theorem by the principle of finite induction.

□

**Theorem 2.** *The generating function of the Tetrarrin sequence for $n \in \mathbb{N}$ is given by:*

$$g(R_{4(n)}, x) = \frac{3 - x^2}{1 - x^2 - x^3 - x^4}.$$

*Proof.* Multiplying the function $g(R_{4(n)}, x)$ by $x^2$,$x^3$ and $x^4$, according to the recurrence formula ($R_{4(n)} = R_{4(n-2)} + R_{4(n-3)} + R_{4(n-4)}$), where $R_{4(0)} = 3, R_{4(1)} = 0, R_{4(2)} = 2, R_{4(3)} = 3$, we have:

$$g(R_{4(n)},x) = R_{4(0)} + R_{4(1)}x + R_{4(2)}x^2 + R_{4(3)}x^3 + R_{4(4)}x^4 + \dots \quad (1)$$

$$x^2 g(R_{4(n)},x) = R_{4(0)}x^2 + R_{4(1)}x^3 + R_{4(2)}x^4 + R_{4(3)}x^5 + R_{4(4)}x^6 + \dots \quad (2)$$

$$x^3 g(R_{4(n)},x) = R_{4(0)}x^3 + R_{4(1)}x^4 + R_{4(2)}x^5 + R_{4(3)}x^6 + R_{4(4)}x^7 + \dots \quad (3)$$

$$x^4 g(R_{4(n)},x) = R_{4(0)}x^4 + R_{4(1)}x^5 + R_{4(2)}x^6 + R_{4(3)}x^7 + R_{4(4)}x^8 + \dots \quad (4)$$

According to the Equation (1-2-3-4), it can be seen that:

$$g(R_{4(n)},x)(1 - x^2 - x^3 - x^4) =^{R}_{4(0)} + R_{4(1)}x + (R_{4(2)} -_4 R_{(0)})x^2 + (R_{4(3)} - R_{4(1)} - R_{4(0)})x^3$$

$$g(R_{4(n)},x)(1 - x^2 - x^3 - x^4) = 3 - x^2$$

$$g(R_{4(n)},x) = \frac{3 - x^2}{1 - x^2 - x^3 - x^4}.$$

□

**Theorem 3.** *The characteristic polynomial of Tetrarrin is:*

$$x^4 - x^2 - x - 1 = 0.$$

*Proof.* Using the Cayley-Hamilton Theorem, we have that the Tetrarrin characteristic polynomial given by [7]:

$$p(\lambda) = det(\lambda I - \Gamma),$$

with $\lambda \in \mathbb{Z}$ or $\lambda \in \mathbb{C}$, $\lambda I = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$ and $\Gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Soon:

$$p(\lambda) = det\left(\begin{bmatrix} \lambda & -1 & 0 & 0 \\ -1 & \lambda & -1 & 0 \\ -1 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{bmatrix}\right) = \lambda^4 - \lambda^2 - \lambda - 1.$$

Then: $p(\lambda) = 0$, we have to: $\lambda^4 - \lambda^2 - \lambda - 1 = 0$.

Soon: $x^4 - x^2 - x - 1 = 0$ □

In which performing the calculation to find the roots of such a polynomial, we have:

$$S = \sqrt[3]{\frac{29}{54} + \sqrt{\left(\frac{-1}{9}\right)^3 + \left(\frac{29}{54}\right)^2}},$$

$$T = \sqrt[3]{\frac{29}{54} - \sqrt{\left(\frac{-1}{9}\right)^3 + \left(\frac{29}{54}\right)^2}}.$$

In this way, the calculated roots are:

$$\begin{aligned}
x_1 &= \frac{1}{3} + S + P, \\
x_2 &= -1, \\
x_3 &= \frac{1}{3} - \frac{1}{2}(S+T) + \frac{\sqrt{3}}{2}(S-T)i, \\
x_4 &= \frac{1}{3} - \frac{1}{2}(S+T) - \frac{\sqrt{3}}{2}(S-T)i.
\end{aligned}$$

**Theorem 4.** *Tetrarrin's Binet formula, with $n \in \mathbb{Z}$ is described by:*

$$R_{4(n)} = A(x_1)^n + B(x_2)^n + C(x_3)^n + D(x_4)^n,$$

*where $n \in \mathbb{Z}$, $x_1, x_2, x_3$ and $x_4$ are the roots of the characteristic Tetrarrin equation and the coefficients are:*

$$\begin{aligned}
A &= \frac{x_1^2 + x_1 + 1}{2x_1^4 + x_1 + 2}, B = \frac{x_2^2 + x_2 + 1}{2x_2^4 + x_2 + 2}, \\
C &= \frac{x_3^2 + x_3 + 1}{2x_3^4 + x_3 + 2}, D = \frac{x_4^2 + x_4 + 1}{2x_4^4 + x_4 + 2}.
\end{aligned}$$

*Proof.* With the recurrence of the Tetrarrin sequence, known as $R_{4(n)} = R_{4(n-2)} + R_{4(n-3)} + R_{4(n-4)}$ , from their initial values given by: $R_{4(0)} = 3, R_{4(1)} = 0, R_{4(2)} = 2, R_{4(3)} = 3$ and from its characteristic polynomial: $x^4 - x^2 - x - 1 = 0$ with roots $x_1, x_2, x_3, x_4$, we have the system resolution:

$$\begin{cases}
A + B + C + D & = 3 \\
Ax_1 + Bx_2 + Cx_3 + Dx_4 & = 0 \\
Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_4^2 & = 2 \\
Ax_1^3 + Bx_2^3 + Cx_3^3 + Dx_4^3 & = 3
\end{cases}$$

Solving the system, we have:

$$A = \frac{-2x_4 - 2x_2 - 3x_2x_3x_4 - 2x_3 + 3}{x_1^3 - x_1^2x_4 - x_1^2x_2 + x_1x_2x_4 - x_1^2x_3 + x_1x_3x_4 - x_2x_3x_4 + x_1x_2x_3},$$
$$B = \frac{-2x_4 - 2x_1 - 3x_1x_3x_4 - 2x_3 + 3}{x_2^3 - x_2^2x_4 - x_1x_2^2 + x_1x_2x_4 - x_1x_3x_4 - x_2^2x_3 + x_2x_3x_4 + x_1x_2x_3},$$
$$C = \frac{2x_4 + 2x_1 + 3x_1x_2x_4 + 2x_2 - 3}{-x_3^3 + x_3^2x_4 + x_1x_3^2 + x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4 - x_1x_2x_3 + x_2x_3^2},$$
$$D = \frac{-2x_1 - 2x_2 - 3x_1x_2x_3 - 2x_3 + 3}{x_4^3 - x_1x_4^2 - x_2x_4^2 + x_1x_2x_4 - x_3x_4^2 + x_1x_3x_4 + x_2x_3x_4 - x_1x_2x_3}.$$

Using the relations $x_1x_2x_3x_4 = -1$, $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = 1$, it is easy to see that:

$$\begin{aligned}
A &= \frac{-2(x_2 + x_3 + x_4) - \frac{3}{x_1} + 3}{x_1^3 - x_1^2(x_2 + x_3 + x_4) + (x_1x_2x_4 + x_1x_3x_4 + x_1x_2x_3) - x_2x_3x_4} \\
&= \frac{-2(-x_1) - \frac{3}{x_1} + 3}{x_1^3 - x_1^2(-x_1) + (1 - x_2x_3x_4) - x_2x_3x_4}, \\
B &= \frac{-2(x_1 + x_3 + x_4) - \frac{3}{x_2} + 3}{x_2^3 - x_2^2(x_1 + x_3 + x_4) + (x_1x_2x_4 + x_2x_3x_4 + x_1x_2x_3) - x_1x_3x_4} \\
&= \frac{-2(-x_2) - \frac{3}{x_2} + 3}{x_2^3 - x_2^2(-x_2) + (1 - x_1x_3x_4) - x_1x_3x_4}, \\
C &= \frac{-2(x_1 + x_2 + x_4) - \frac{3}{x_3} + 3}{x_3^3 - x_3^2(x_1 + x_2 + x_4) + (x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_3) - x_1x_2x_4} \\
&= \frac{-2(-x_3) - \frac{3}{x_3} + 3}{x_3^3 - x_3^2(-x_3) + (1 - x_1x_2x_4) - x_1x_2x_4}, \\
D &= \frac{-2(x_1 + x_2 + x_3) - \frac{3}{x_4} + 3}{x_4^3 - x_4^2(x_1 + x_2 + x_3) + (x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_4) - x_1x_2x_3} \\
&= \frac{-2(-x_4) - \frac{3}{x_4} + 3}{x_4^3 - x_4^2(-x_4) + (1 - x_1x_2x_3) - x_1x_2x_3}.
\end{aligned}$$

Logo:

$$
\begin{aligned}
A &= \frac{-2(-x_1) - \frac{3}{x_1} + 3}{2x_1^3 + (1 + \frac{1}{x_1}) + \frac{1}{x_1}} = \frac{2x_1 + \frac{3}{x_1} + 3}{2x_1^3 + 1 + \frac{2}{x_1}} = \frac{2x_1^2 + 3x_1 + 3}{2x_1^4 + x_1 + 2},\\
B &= \frac{-2(-x_2) - \frac{3}{x_2} + 3}{2x_2^3 + (1 + \frac{1}{x_2}) + \frac{1}{x_2}} = \frac{2x_2 + \frac{3}{x_2} + 3}{2x_2^3 + 1 + \frac{2}{x_2}} = \frac{2x_2 + \frac{3}{x_2} + 3}{2x_2^3 + 1 + \frac{2}{x_2}} = \frac{2x_2^2 + 3x_2 + 3}{2x_2^4 + x_2 + 2},\\
C &= \frac{-2(-x_3) - \frac{3}{x_3} + 3}{2x_3^3 + (1 + \frac{1}{x_3}) + \frac{1}{x_3}} = \frac{2x_2 + \frac{3}{x_2} + 3}{2x_2^3 + 1 + \frac{2}{x_2}} = \frac{-2x_3 - \frac{3}{x_3} - 3}{-2x_3^3 - 1 - \frac{2}{x_3}} = \frac{2x_3^2 + 3x_3 + 3}{2x_3^4 + x_3 + 2},\\
D &= \frac{-2(-x_4) - \frac{3}{x_4} + 3}{2x_2^4 + (1 + \frac{1}{x_4}) + \frac{1}{x_4}} = \frac{2x_4 + \frac{3}{x_4} + 3}{2x_2^4 + 1 + \frac{2}{x_4}} = \frac{2x_4 + \frac{3}{x_4} + 3}{2x_4^3 + 1 + \frac{2}{x_4}} = \frac{2x_4^2 + 3x_4 + 3}{2x_4^4 + x_4 + 2}.
\end{aligned}
$$

□

In this way, the study of the generalization of the Tetrarrin sequence is carried out, extending these numbers to non-positive integer indices. Thus, from the recurrence of Tetrarrin, we have

Table 1: Tetrarrin Sequence Terms - Non - Positive Integer Index. Source: Prepared by the authors

| $R_{4(-10)}$ | $R_{4(-9)}$ | $R_{4(-8)}$ | $R_{4(-7)}$ | $R_{4(-6)}$ | $R_{4(-5)}$ | $R_{4(-4)}$ | $R_{4(-3)}$ | $R_{4(-2)}$ | $R_{4(-1)}$ | $R_{4(0)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 9 | -7 | -2 | 5 | 0 | -4 | 3 | 1 | -1 | 0 | 3 |

**Theorem 5.** *Let $n \in \mathbb{N}$, the matrix form of the Tetrarrin sequence for non-positive terms is given by:*

$$
\boldsymbol{v_t}\delta^n = \begin{bmatrix} 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} R_{4(-n)} & R_{4(-n+1)} & R_{4(-n+2)} & R_{4(-n+3)} \end{bmatrix},
$$

*where $\boldsymbol{v_t}$ is the row vector containing the initial values of the Tetrarrin sequence and $\delta = \Gamma^{-1}$.*

*Proof.* Proving that for $n = 1$, the equality is true, we have:

$$\begin{aligned}
\boldsymbol{v_t}\delta &= \begin{bmatrix} 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 3 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} R_{4(-1)} & R_{4(0)} & R_{4(1)} & R_{4(2)} \end{bmatrix}.
\end{aligned}$$

Assuming it holds for any $n = k, k \in \mathbb{N}$:

$$\begin{aligned}
\boldsymbol{v_t}\delta^k &= \begin{bmatrix} 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k \\
&= \begin{bmatrix} R_{4(-k)} & R_{4(-k+1)} & R_{4(-k+2)} & R_{4(-k+3)} \end{bmatrix}.
\end{aligned}$$

Then, verifying that it also holds for $n = k + 1$:

$$\begin{aligned}
\boldsymbol{v_t}\delta^{k+1} &= \boldsymbol{v_t}\delta^k \\
&= \begin{bmatrix} 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R_{4(-k)} & R_{4(-k+1)} & R_{4(-k+2)} & R_{4(-k+3)} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -R_{4(-k)} - R_{4(-k+1)} + R_{4(-k+3)} & R_{4(-k)} & R_{4(-k+1)} & R_{4(-k+2)} \end{bmatrix} \\
&= \begin{bmatrix} R_{4(-k-1)} & R_{4(-k)} & R_{4(-k+1)} & R_{4(-k+2)} \end{bmatrix}.
\end{aligned}$$

Thus validating the theorem by the principle of finite induction.

□

**Theorem 6.** *The generating function of the Tetrarrin sequence for non-positive terms is* *given by:*

$$g(R_{4(n)}, x) = \frac{x^3 - 3x^2 - 3x}{1 - x^2 - x^3 - x^4}.$$

*Proof.* Based on the function $g(R_{4(-n)}, x) = R_{4(0)} + R_{4(-1)}x + R_{4(-2)}x^2 + R_{4(-3)}x^3 + R_{4(-4)}x^4 + \ldots$, it can be multiplied by $x$, $x^2$, $x^4$ obtaining:

$$g(R_{4(-n)}, x) = R_{4(0)} + R_{4(-1)}x + R_{4(-2)}x^2 + R_{4(-3)}x^3 + R_{4(-4)}x^4 + \ldots \quad (5)$$

$$xg(R_{4(-n)}, x) = R_{4(0)}x + R_{4(-1)}x^2 + R_{4(-2)}x^3 + R_{4(-3)}x^4 + R_{4(-4)}x^5 + \ldots \quad (6)$$

$$x^2g(R_{4(-n)}, x) = R_{4(0)}x^2 + R_{4(-1)}x^3 + R_{4(-2)}x^4 + R_{4(-3)}x^5 + R_{4(-4)}x^6 + \ldots \quad (7)$$

$$x^4g(R_{4(-n)}, x) = R_{4(0)}x^4 + R_{4(-1)}x^5 + R_{4(-2)}x^6 + R_{4(-3)}x^7 + R_{4(-4)}x^8 + \ldots \quad (8)$$

Thus performing 8 − [7 + 6 + 5], we have to $x^4g(R_{4(-n)}, x) - [x^2g(R_{4(-n)}, x) + xg(R_{4(-n)}, x) + g(R_{4(-n)}, x)]$ and with $R_{4(0)} = 3, R_{4(-1)} = 0, R_{4(-2)} = -1$, where:

$$g(R_{4(-n)}, x)(x^4 - x^2 - x - 1) = -R_{4(0)}(x^2 + x) - R_{4(-1)}(x^3 + x^2) - (R_{4(-2)}x^3)$$

$$g(R_{4(-n)}, x)(x^4 - x^2 - x - 1) = x^3 - 3x^2 - 3x$$

$$g(R_{4(-n)}, x) = \frac{x^3 - 3x^2 - 3x}{(x^4 - x^2 - x - 1)}.$$

□

# 3 Conclusions

The present study carried out a definition of a new sequence, called the Tetrarin sequence. From the Perrin sequence, it was possible to define the numbers discussed in this work. With this, mathematical studies were carried out, involving the Binet formula, matrix form and generating function for this fourth order sequence.

For future studies, new ideas and mathematical properties are sought, deepening the investigation of these numbers in the area of recurrent numerical sequences.

## Thanks

The first author thanks the Fundação Cearense de Apoio ao Desenvolvimento Científico e Tecnológico (Funcap) for its support. The second author acknowledges the support of the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and the third author expresses gratitude to FCT - Fundação para a Ciência e Tecnologia, I. P., under the scope of the project UID / CED / 00194/2020.

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Submitted in 15 Sept. 2023
Accepted in 02 Aug. 2024