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# Operational properties involving functions with generalized continuity

## Propriedades operacionais envolvendo funções com continuidade generalizada

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**Abstract**. In the work [11], Vieira introduced a new perspective about continuity of functions, which involves the idea of a suitable type of continuity of a function with respect to another function. The inspiration for this notion of generalized continuity arises naturally from the concept of generalized limit of a function with respect to another function, thus expanding the field of mathematical knowledge about continuity of functions. Initially presented by Vieira and Braz in [1], the concept of generalized limit is relevant, since a Riemann integral is a case of a generalized limit, as can be seen in [11]. This article proposes to further explore this notion of generalized continuity and its main focus is to investigate and present operational properties that arise when we deal with functions that exhibit this generalized continuity, operational properties such as composition, concatenation, among others. Through this study, it is hoped to shed light on the meaning and implications of these properties in this context of generalized continuity, allowing a broader understanding of this notion of continuity.

**Keywords**. Generalized continuity. Continuator. Continuant.

**Resumo**. No trabalho [11], Vieira introduziu uma nova perspectiva sobre continuidade de funções, a qual envolve a ideia de um tipo apropriado de continuidade de uma função com respeito a outra função. A inspiração para essa noção de continuidade generalizada surge naturalmente do conceito de limite generalizado de uma função com respeito a outra função, expandindo assim o campo do conhecimento matemático sobre continuidade de funções. Apresentado inicialmente por Vieira e Braz em [1], o conceito de limite generalizado possui relevância, uma vez que uma integral de Riemann é um caso de limite generalizado, como pode ser visto em [11]. Este artigo se propõe a explorar mais essa noção de continuidade generalizada e o foco principal dele é investigar e apresentar propriedades operacionais que surgem quando lidamos com funções que exibem essa continuidade generalizada, propriedades operacionais tais como composição, concatenação, entre outras. Através deste estudo, espera-se lançar luz sobre o significado e as implicações dessas propriedades neste contexto de continuidade generalizada, permitindo uma compreensão mais ampla sobre esta noção de continuidade.

**Palavras-chave**. Continuidade generalizada. Continuador. Continuante.

**Mathematics Subject Classification (MSC):** primary 54C08; secondary 54C10, 54D05, 54D30, 93C30.

# 1 Introduction

In [11], Vieira introduced the notion of generalized continuity of a function with respect to another function based on the concept of generalized limit. One of the motivations for introducing the concept of a generalized limit of a function with respect to another function is that a Riemann integral of a function $f$ on an interval $[a, b]$ is properly defined as a generalized limit. More precisely, it is defined as a generalized limit of the function given by the Riemann sum of $f$ with respect the function given by the partition norm of $[a, b]$. Furthermore, the concept of a generalized limit in fact generalizes the concept of a usual limit of a function, since a usual limit of a function $f$ with domain $X$ is a generalized limit of the function $f$ with respect to the identity function defined in $X$.

The term generalized continuity has been approached and studied more frequently in the mathematical literature in recent decades. Császár in 2002 introduced the notion of generalized topology in [2], which differs from the notion of topology for lacking the property about finite intersection of open sets. From this notion many types of generalized continuity can be defined in these generalized topological spaces, for example, $(g, g')$-continuity [2], $\theta(g, g')$-continuity [2] and $gm$-continuity [12]. In the year 2000, Popa and Noiri introduced in [10] the concept of minimal structure and from that concept they introduced a version of generalized continuity called $m$-continuous functions [9, 10].

The generalized continuities defined from the notions given by Császár, Popa and Noiri, do not coincide with the generalized continuity introduced by Vieira. The functions with continuities based on the notions given by the three authors above have domains and codomains with more general structures than topologies (such as generalized topologies or minimal structures) and have continuity rules defined from the opens of these structures. On the other hand, the notion of continuity addressed by Vieira does not require topological structures in the domains of the functions, only in their codomains and yet such continuity is relative to another function called *continuator*. Another notion of generalized continuity present in scientific works, and distinct from the notion approached by Vieira, is the continuity presented by Kupka ([4], [5], [6] and [7]), as can be seen in [11].

Considering the concept of generalized continuity introduced by Vieira, the authors of this article asked whether classical properties of compositions, concatenations and other properties involving continuous functions would have similar versions in this context of generalized continuity. This article aims to present some operational properties for functions that have generalized continuity in the sense given by Vieira, similar to those properties involving continuous functions in the usual sense.

# 2 Preliminaries

In this article, the notation $\mathcal{P}(X)$ indicates the collection of all subsets of the set X. The notation $(X, \mathcal{T}^X)$ indicates that the set $X$ is equipped with the topology $\mathcal{T}^X$, that is, it denotes a topological space. Considering a topological space $(X, \mathcal{T}^X)$ and a subset $A$ of $X$, the set

$$\mathcal{T}_A^X = \{U \in \mathcal{T}^X : \; A \subset U\}$$

denotes the collection of all open sets of the topological space $(X, \mathcal{T}^X)$ that contains $A$. If $A = \{a\}$, then the collection $\mathcal{T}_A^X$ is simply denoted by $\mathcal{T}_a^X$. The set of all functions from a set $A$ to a topological space $(B, \mathcal{T}^B)$ is denoted by $\mathsf{F}(A, (B, \mathcal{T}^B))$ or simply by $\mathsf{F}(A, B)$, when it is clear which topology is adopted in the codomain $B$.

**Definition 1.** [11, p. 86] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f \colon X \to (Y, \mathcal{T}^Y)$ and $n \colon X \to (Z, \mathcal{T}^Z)$ be functions and $a \in X$. The function $f$ is said to be $n$-*continuous *at $a$ if for every $U \in \mathcal{T}_{f(a)}^Y$, there exists a $V \in \mathcal{T}_{n(a)}^Z$ such that*

$$f(n^{-1}(V)) \subset U \,. \tag{1}$$

A function $f$ that is $n$-continuous at $a$ can also be called a *continuant* of $n$ at $a$ and the function $n$ is called a *continuator* of $f$ at $a$. Fixing a function $n \colon X \to (Z, \mathcal{T}^Z)$, a point

$a \in X$ and a topological space $(Y, \mathcal{T}^Y)$, the set

$$\mathsf{C}_n(a, X, Y) = \{f \in \mathsf{F}(X, (Y, \mathcal{T}^Y)) : \ f \text{ is } n\text{-continuous at } a\} \tag{2}$$

represents the set of all continuants of $n$ at $a$ with respect to $X$ and $(Y, \mathcal{T}^Y)$. An element of the set

$$\mathsf{C}_n(X, Y) = \bigcap_{a \in X} \mathsf{C}_n(a, X, Y) \tag{3}$$

is an $n$-continuous function at all points in $X$. The set $\mathsf{C}_n(X, Y)$ represents the set of all continuants of $n$ with respect to $X$ and $(Y, \mathcal{T}^Y)$. On the other hand, fixing a function $f \colon X \to (Y, \mathcal{T}^Y)$, a point $a \in X$ and a topological space $(Z, \mathcal{T}^Z)$, the set

$$\mathbf{C}^f(a, X, Z) = \{n \in \mathsf{F}(X, (Z, \mathcal{T}^Z)) : \ f \in \mathsf{C}_n(a, X, Y)\} \tag{4}$$

represents the set of all continuators of $f$ at $a$ with respect to $X$ and $(Z, \mathcal{T}^Z)$. The set

$$\mathbf{C}^f(X, Z) = \bigcap_{a \in X} \mathbf{C}^f(a, X, Z) = \{n \in \mathsf{F}(X, (Z, \mathcal{T}^Z)) : \ f \in \mathsf{C}_n(X, Y)\} \tag{5}$$

represents the set of all continuators of $f$ with respect to $X$ and $(Y, \mathcal{T}^Y)$.

It can be noted that in (2), (3), (4) and (5) the topologies in the codomains of the functions are omitted in the notations of this sets, since this information is clear in the context.

**Example 1.** [11, p. 87] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ be a topological space and let $a \in X$. Every function $f \colon X \to (Y, \mathcal{T}^Y)$ is $f$-continuous at $a$. In fact, given $U \in \mathcal{T}^Y_{f(a)}$, then take $V = U$ and it follows that*

$$f(f^{-1}(V)) = f(f^{-1}(U)) \subset U.$$

*Therefore, $f$ is $f$-continuous at $a$, that is, $f \in \mathsf{C}_f(a, X, Y)$ and $f \in \mathbf{C}^f(a, X, Y)$.*

**Example 2.** [11, p. 88] *Let $(X, \mathcal{T}^X)$ and $(Y, \mathcal{T}^Y)$ be topological spaces, $a \in X$ and $id \colon X \to (X, \mathcal{T}^X)$ be the identity function. A function $f \colon (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is continuous at $a$ if, and only if, $f \colon X \to (Y, \mathcal{T}^Y)$ is $id$- continuous at $a$. In fact, when $V \in \mathcal{T}^X_a$ is considered, note that $f(id^{-1}(V)) = f(V)$. Therefore, the condition required in the classical definition of continuity and the condition (1) are equivalent.*

The previous example shows that classical continuity is a particular case of generalized continuity, when the adopted continuator is the identity function.

On the other hand, it is possible to characterize generalized continuity by providing the domain of the continuant with an appropriate topology, as can be checked below. Let $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions and let $\mathcal{T}_n = \{n^{-1}(V)\colon V \in \mathcal{T}^Z\}$ be the induced topology by $n$ on $X$. The function $f\colon X \to (Y, \mathcal{T}^Y)$ is $n$-continuous if, and only if, the function $f\colon (X, \mathcal{T}_n) \to (Y, \mathcal{T}^Y)$ is continuous in the usual sense.

An important aspect to highlight is that the continuity generalized in the sense introduced by Vieira does not require the provision of the domain of the continuant with a topology. In addition, the definition 1 of generalized continuity adopted in this article arises naturally from the concept of generalized limit, which has theoretical relevance. For example, a Riemann integral is a generalized limit, as can be seen at [11]. For these reasons, the authors of this work prefer to address generalized continuity as defined in 1.

**Example 3.** [11, p. 88] *Let $\mathbb{R}$ be equipped with the usual topology, $a$ be a real number greater than $0$ and let $id\colon \mathbb{R} \to \mathbb{R}$ be the identity function and $m\colon \mathbb{R} \to \mathbb{R}$ the function given by $m(x) = |x|$. For every open subset $V$ of $\mathbb{R}$ containing $m(a) = a$, consider $V^+ = V \cap [0, \infty)$ and $V^- = \{x \in \mathbb{R} :\ -x \in V^+\}$. Note that $m^{-1}(V) = V^+ \cup V^-$. Considering the open subset $U = (0, \infty)$ of $\mathbb{R}$ containing $id(a) = a$, it results that for every open subset $V$ of $\mathbb{R}$ containing $m(a) = a$ is valid that*

$$id(m^{-1}(V)) = id(V^+ \cup V^-) = V^+ \cup V^- \not\subset (0, \infty) = U.$$

*Therefore, $id$ is not $m$-continuous at $a$, that is, $id \notin \mathsf{C}_m(a, \mathbb{R}, \mathbb{R})$ and $m \notin \mathbf{C}^{id}(a, \mathbb{R}, \mathbb{R})$.*

**Proposition 1.** [11, p. 86] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions and $a \in X$. The following statements hold true:*

(i) *If $f$ is constant function, then $f \in \mathsf{C}_n(a, X, Y)$.*

(ii) *If $n$ is constant function and $f \in \mathsf{C}_n(a, X, Y)$, then $f(X) \subset U$, for all $U \in \mathcal{T}^Y_{f(a)}$.*

**Theorem 1.** [11, p. 96] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$, $(Z, \mathcal{T}^Z)$ and $(R, \mathcal{T}^R)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$, $m\colon X \to (Z, \mathcal{T}^Z)$ and $n\colon X \to (R, \mathcal{T}^R)$ be functions and $a \in X$. If $f \in \mathsf{C}_m(a, X, Y)$ and $m \in \mathsf{C}_n(a, X, Z)$, then $f \in \mathsf{C}_n(a, X, Y)$.*

**Definition 2.** [11, p. 89] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions and $a \in X$. The function $f$ is said to be* widely $n$- continuous *at $a$ if for every $U \in \mathcal{T}^Y_{f(a)}$, there exists $V \in \mathcal{T}^Z_{n(a)}$ such that*

$$V \cap n(X) \subset n(f^{-1}(U))\,. \tag{6}$$

A function $f$ that is widely $n$-continuous at $a$ can also be called a *wide continuant* of $n$ at $a$ and the function $n$ is called a *wide continuator* of $f$ at $a$. Fixing a function $n\colon X \to (Z, \mathcal{T}^Z)$, a point $a \in X$ and a topological space $(Y, \mathcal{T}^Y)$, the set

$$\mathsf{W}_n(a, X, Y) = \{f \in \mathsf{F}(X, (Y, \mathcal{T}^Y)) :\ f \text{ is widely } n\text{-continuous at } a\} \tag{7}$$

represents the set of all continuants of $n$ at $a$ with respect to $X$ and $(Y, \mathcal{T}^Y)$. An element of the set

$$\mathsf{W}_n(X, Y) = \bigcap_{a \in X} \mathsf{W}_n(a, X, Y) \tag{8}$$

is a widely $n$-continuous function at all points in $X$. The set $\mathsf{W}_n(X, Y)$ represents the set of all wide continuants of $n$ with respect to $X$ and $(Y, \mathcal{T}^Y)$. On the other hand, fixing a function $f\colon X \to (Y, \mathcal{T}^Y)$, a point $a \in X$ and a topological space $(Z, \mathcal{T}^Z)$, the set

$$\mathbf{W}^f(a, X, Z) = \{n \in \mathsf{F}(X, (Z, \mathcal{T}^Z)) :\ f \in \mathsf{W}_n(a, X, Y)\} \tag{9}$$

represents the set of all wide continuators of $f$ at $a$ with respect to $X$ and $(Z, \mathcal{T}^Z)$. The set

$$\mathbf{W}^f(X, Z) = \bigcap_{a \in X} \mathbf{W}^f(a, X, Z) \tag{10}$$

represents the set of all wide continuators of $f$ with respect to $X$ and $(Y, \mathcal{T}^Y)$.

**Example 4.** [11, p. 90] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ be topological space and $a \in X$. Every function $f\colon X \to (Y, \mathcal{T}^Y)$ is widely $f$-continuous at $a$. In fact, given $U \in \mathcal{T}^Y_{f(a)}$, then take $V = U$ and it follows that*

$$V \cap f(X) = U \cap f(X) = f\left(f^{-1}(U) \cap X\right) = f\left(f^{-1}(U)\right).$$

*Therefore, $f$ is widely $f$-continuous at $a$, that is, $f \in \mathsf{W}_f(a, X, Y)$ and $f \in \mathbf{W}^f(a, X, Y)$.*

**Example 5.** [11, p. 90] *Let $\mathbb{R}$ be equipped with usual topology, $a$ be a real number greater than $0$, $id\colon \mathbb{R} \to \mathbb{R}$ be the identity function and $m\colon \mathbb{R} \to \mathbb{R}$ be the function given by $m(x) = |x|$. Given $U$ a open subset of $\mathbb{R}$ containing $id(a) = a$, then $V = U \cap (0, \infty)$ is an open subset of $\mathbb{R}$ containing $m(a) = a$ and $V \subset m(U)$. Thus, it is valid that*

$$V \cap m(\mathbb{R}) = V \cap [0, \infty) = V \subset m(U) = m\left(id^{-1}(U)\right).$$

*Hence, $id$ is widely $m$-continuous at $a$, that is, $id \in \mathsf{W}_m(a, \mathbb{R}, \mathbb{R})$ and $m \in \mathbf{W}^{id}(a, \mathbb{R}, \mathbb{R})$.*

**Theorem 2.** [11, p. 91] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological* *spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions and $a \in X$. If $f$ is $n$-continuous at $a$, then $f$ is widely $n$-continuous at $a$.*

**Corollary 1.** [11, p. 92] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions and $a \in X$. If $f$ is a constant function, then $f \in \mathsf{W}_n(a, X, Y)$.*

Consider $X$ a non-empty set, $(Y, \mathcal{T}^Y)$ a topological space and $n\colon X \to (Z, \mathcal{T}^Z)$ a continuator, then $\mathcal{T}^{n(X)}$ denotes the induced topology by $\mathcal{T}^Z$ on $n(X)$.

**Theorem 3.** [11, p. 96] *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions. The function $f$ is widely $n$-continuous if, and only if, $n(f^{-1}(U)) \in \mathcal{T}^{n(X)}$, for all $U \in \mathcal{T}^Y$.*

# 3 Generalized continuity equivalences

This section aims to present some additional results about generalized continuity. In this article, given a topological space $(X, \mathcal{T}^X)$ and $a \in X$, a subset $U$ of $X$ is said to be a *neighborhood* of $a$ if this point belongs to the interior of $U$, that is, $a \in U^\circ$. The symbol $\mathcal{V}_a^X$ denotes the collection of all neighborhoods of $a$ with respect to $\mathcal{T}^X$. A subcollection $\mathcal{B}_a^X$ of $\mathcal{V}_a^X$ (i.e., $\mathcal{B}_a^X \subset \mathcal{V}_a^X$) is a *basis of neighborhoods* of $a$ with respect to $\mathcal{T}^X$ if, and only if, for each $U \in \mathcal{V}_a^X$, there exists $W \in \mathcal{B}_a^X$ such that $W \subset U$. It is easily verified that if $\mathcal{B}_a^X$ is a basis of neighborhoods of $a$ with respect to $\mathcal{T}^X$ and $U, V \in \mathcal{B}_a^X$, then there exists $W \in \mathcal{B}_a^X$ such that $W \subset U \cap V$.

**Proposition 2.** *Let $X$ be a non-empty set, $a \in X$, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions, $\mathcal{B}_{f(a)}^Y$ be a basis neighborhoods of $f(a)$ with respect to $\mathcal{T}^Y$ and $\mathcal{B}_{n(a)}^Z$ be a basis neighborhoods of $n(a)$ with respect to $\mathcal{T}^Z$. The following statements are equivalent:*

(i) $f \in \mathsf{C}_n(a, X, Y)$.

(ii) *For every $U \in \mathcal{V}_{f(a)}^Y$, there exists $V \in \mathcal{V}_{n(a)}^Z$ such that $f(n^{-1}(V)) \subset U$.*

(iii) *For every $U \in \mathcal{B}_{f(a)}^Y$, there exists $V \in \mathcal{B}_{n(a)}^Z$ such that $f(n^{-1}(V)) \subset U$.*

*Proof.* $\boxed{\text{(i)} \Rightarrow \text{(ii)}}$ If $U \in \mathcal{V}_{f(a)}^Y$, then $f(a) \in U^\circ$. As $U^\circ \in \mathcal{T}_{f(a)}^Y$, then there exists $V_1 \in \mathcal{T}_{n(a)}^Z$ such that $f(n^{-1}(V_1)) \subset U^\circ$. Take $V = V_1$, which implies that $V \in V_{n(a)}^Z$ and it is obtained that

$$f(n^{-1}(V)) \subset U^\circ \subset U.$$

(ii) ⇒ (iii) Let $U \in \mathcal{B}^Y_{f(a)}$. As $U \in \mathcal{V}^Y_{f(a)}$, then by hypothesis there exists $V_1 \in \mathcal{V}^Z_{n(a)}$ such that $f(n^{-1}(V_1)) \subset U$. As $V_1 \in \mathcal{V}^Z_{n(a)}$, then by definition of neighborhood basis of the point $n(a)$, exists $V \in \mathcal{B}^Z_{n(a)}$ such that $V \subset V_1$. Therefore,

$$f(n^{-1}(V)) \subset f(n^{-1}(V_1)) \subset U.$$

(iii) ⇒ (i) If $U \in \mathcal{T}^Y_{f(a)}$, then $U \in \mathcal{V}^Y_{f(a)}$ and therefore there exists $W \in \mathcal{B}^Y_{f(a)}$ such that $W \subset U$. By hypothesis there exists $V_1 \in \mathcal{B}^Z_{n(a)}$ such that $f(n^{-1}(V_1)) \subset W \subset U$. Take $V = V_1^\circ \in \mathcal{T}^Z_{n(a)}$. As $V \subset V_1$ it follows that

$$f(n^{-1}(V)) \subset f(n^{-1}(V_1)) \subset W \subset U.$$

□

**Proposition 3.** *Let $X$ be a non-empty set, $a \in X$, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions, $\mathcal{B}^Y_{f(a)}$ be a basis neighborhoods of $f(a)$ with respect to $\mathcal{T}^Y$ and $\mathcal{B}^Z_{n(a)}$ be a basis neighborhoods of $n(a)$ with respect to $\mathcal{T}^Z$. The following statements are equivalent:*

(i) $f \in \mathsf{W}_n(a, X, Y)$.

(ii) *For every $U \in \mathcal{V}^Y_{f(a)}$, there exists $V \in \mathcal{V}^Z_{n(a)}$ such that $V \cap n(X) \subset n(f^{-1}(U))$.*

(iii) *For every $U \in \mathcal{B}^Y_{f(a)}$, there exists $V \in \mathcal{B}^Z_{n(a)}$ such that $V \cap n(X) \subset n(f^{-1}(U))$.*

*Proof.* (i) ⇒ (ii) If $U \in \mathcal{V}^Y_{f(a)}$, then $f(a) \in U^\circ$ and $U^\circ \in \mathcal{T}^Y_{f(a)}$. By hypothesis, for $U^\circ$ there exists $W \in \mathcal{T}^Z_{n(a)}$ such that

$$W \cap n(X) \subset n(f^{-1}(U^\circ)) \subset n(f^{-1}(U)).$$

As $n(a) \in W$ and $W = W^\circ$, then $W \in \mathcal{V}^Z_{n(a)}$. Take $V = W$. It follows that

$$V \cap n(X) \subset n(f^{-1}(U)).$$

(ii) ⇒ (iii) If $U \in \mathcal{B}^Y_{f(a)}$, then $U \in \mathcal{V}^Y_{f(a)}$. By hypothesis, for $U$ there exists $W \in \mathcal{V}^Z_{n(a)}$ such that $W \cap n(X) \subset n(f^{-1}(U))$. As $W \in \mathcal{V}^Z_{n(a)}$, then there exists $V \in \mathcal{B}^Z_{n(a)}$ such that $V \subset W$. Therefore, $V \in \mathcal{B}^Z_{n(a)}$ satisfies

$$V \cap n(X) \subset W \cap n(X) \subset n(f^{-1}(U)).$$

(iii) ⇒ (i) If $U \in \mathcal{T}^Y_{f(a)}$, then $U \in \mathcal{V}^Y_{f(a)}$. Hence, there exists $W_1 \in \mathcal{B}^Y_{f(a)}$ such that $W_1 \subset U$. By hypothesis, there exists $W_2 \in \mathcal{B}^Z_{n(a)}$ such that $W_2 \cap n(X) \subset n(f^{-1}(W_1))$.

Take $V = W_2^\circ \in \mathcal{T}_{n(a)}^Z$. It follows that

$$V \cap n(X) = W_2^\circ \cap n(X) \subset W_2 \cap n(X) \subset n(f^{-1}(W_1)) \subset n(f^{-1}(U)).$$

□

**Proposition 4.** *Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $n\colon X \to (Z, \mathcal{T}^Z)$ be an injective function, $\mathcal{F}^Y$ be the collection of all closed sets with respect to $\mathcal{T}^Y$ and $\mathcal{F}^{n(X)}$ be the collection of all closed sets with respect to $\mathcal{T}^{n(X)}$. It holds that $f \in \mathsf{W}_n(X, Y)$ if, and only if, $n\left(f^{-1}(F)\right) \in \mathcal{F}^{n(X)}$, for all $F \in \mathcal{F}^Y$*

*Proof.* Let $F \in \mathcal{F}^Y$, then $Y - F \in \mathcal{T}^Y$. As $f \in \mathsf{W}_n(X, Y)$, it follows from Theorem 3 that $n\left(f^{-1}(Y - F)\right) \in \mathcal{T}^{n(X)}$. Note that since $n$ is injective, it follows that $n\left(f^{-1}(Y)\right) - n\left(f^{-1}(F)\right) = n\left(f^{-1}(Y) - f^{-1}(F)\right)$. As $n(X) = n\left(f^{-1}(Y)\right)$, then

$$n\left(X\right) - n\left(f^{-1}(F)\right) = n\left(f^{-1}(Y) - f^{-1}(F)\right) = n\left(f^{-1}(Y - F)\right) . \tag{11}$$

Therefore, $n\left(f^{-1}(F)\right) \in \mathcal{F}^{n(X)}$.
Conversely, let $U \in \mathcal{T}^Y$, then $Y - U \in \mathcal{F}^Y$. By hypothesis $n\left(f^{-1}(Y - U)\right) \in \mathcal{F}^{n(X)}$ and $n$ is injective, so it follows similarly to (11) that $n(X) - n\left(f^{-1}(U)\right) \in \mathcal{F}^{n(X)}$. Therefore, $n\left(f^{-1}(U)\right) \in \mathcal{T}^{n(X)}$ and it follows from Theorem 3 that $f \in \mathsf{W}_n(X, Y)$. □

Let $(X, \mathcal{T}^X)$ be a topological space. A subcollection $\mathcal{B}$ of $\mathcal{T}^X$ (i.e., $\mathcal{B} \subset \mathcal{T}^X$) is a *basis* of the topology $\mathcal{T}^X$ if given $U \in \mathcal{T}^X$, there exists $\mathcal{C} \subset \mathcal{B}$ such that

$$U = \bigcup_{V \in \mathcal{C}} V .$$

An element of a basis $\mathcal{B}$ of $\mathcal{T}^X$ is called *basic open* with respect to $\mathcal{T}^X$ and the set

$$\mathcal{B}_a = \{U \in \mathcal{B} : \ a \in U\}$$

denotes the collection of all basic opens around the point $a \in X$. It is easy to verify that $\mathcal{B}_a$ is a basis of neighborhoods for $a$ with respect to $\mathcal{T}^X$.

**Example 6.** *The set $\mathcal{R} = \{(a, b) \colon a, b \in \mathbb{R}\}$ is a basis of the usual topology $\mathcal{T}^{\mathbb{R}}$ on $\mathbb{R}$.*

Already a subcollection $\mathcal{S}$ of $\mathcal{T}^X$ (i.e., $\mathcal{S} \subset \mathcal{T}^X$)is a *subbasis* of the topology $\mathcal{T}^X$ if given $U \in \mathcal{T}^X$, there exists an index set $\Lambda$ and for each $\lambda \in \Lambda$, there exists a finite subcolletion $\mathcal{C}_\lambda \subset \mathcal{S}$ such that

$$U = \bigcup_{\lambda \in \Lambda} \left( \bigcap_{V \in \mathcal{C}_\lambda} V \right) .$$

In other words, $\mathcal{S}$ is a subbasis of the topology $\mathcal{T}^X$ if given an open $U \in \mathcal{T}^X$, it is written as an arbitrary union of finite intersections of members of $\mathcal{S}$. An element of a subbasis $\mathcal{S}$ of $\mathcal{T}^X$ is called a *subbasic open subset* with respect to $\mathcal{T}^X$ and the set

$$\mathcal{S}_a = \{U \in \mathcal{S} : \ a \in U\}$$

denotes the collection of all subbasic open subsets around the point $a \in X$. Note that every basis of a topology is also a subbasis.

**Example 7.** *The set* $\mathcal{S} = \{(a, \infty) \colon a \in \mathbb{R}\} \cup \{(-\infty, b) \colon b \in \mathbb{R}\}$ *is a subbasis of the usual topology* $\mathcal{T}^{\mathbb{R}}$ *on* $\mathbb{R}$.

Let $Y$ be a non-empty set and $\mathcal{R}$ be a collection of subsets of $Y$. Consider the following collections of subsets of $Y$ given by

$$[\mathcal{R}] = \{U \in \mathcal{P}(Y) \ : \ U = \bigcap_{i=1}^{n} V_i \text{ for some } n \in \mathbb{N} \text{ and each } V_i \in \mathcal{R}\} \tag{12}$$

and

$$\langle\mathcal{R}\rangle = \{U \in \mathcal{P}(Y) \ : \ U = \bigcup_{V \in \mathcal{D}} V \text{ for some } \mathcal{D} \subset \mathcal{R}\}. \tag{13}$$

It is known that if $Y = \bigcup_{V \in \mathcal{R}} V$, then the collection $\langle[\mathcal{R}]\rangle$ is a topology on $Y$ called *topology generated by* $\mathcal{R}$ and it is denoted by $\langle\!\langle\mathcal{R}\rangle\!\rangle$. Furthermore, the collection $[\mathcal{R}]$ is a basis of the topology $\langle\!\langle\mathcal{R}\rangle\!\rangle$ called *basis generated by* $\mathcal{R}$ or *generator basis* of $\langle\!\langle\mathcal{R}\rangle\!\rangle$ and $\mathcal{R}$ is a subbasis of the topology $\langle\!\langle\mathcal{R}\rangle\!\rangle$ called *generator subbasis* of $\langle\!\langle\mathcal{R}\rangle\!\rangle$. In the case that $(X, \mathcal{T}^X)$ is a topological space and $\mathcal{S}$ is a subbasis of $\mathcal{T}^X$, then $\langle\!\langle\mathcal{S}\rangle\!\rangle = \mathcal{T}^X$. Furthermore, if $a \in X$, then $[\mathcal{S}]_a$ is a basis of neighborhoods for $a$ with respect to $\mathcal{T}^X$.

**Theorem 4.** *Let* $X$ *be a non-empty set,* $a \in X$*,* $(Y, \mathcal{T}^Y)$ *and* $(Z, \mathcal{T}^Z)$ *be topological spaces,* $f \colon X \to (Y, \mathcal{T}^Y)$ *and* $n \colon X \to (Z, \mathcal{T}^Z)$ *be functions,* $\mathcal{R}$ *be a subbasis of* $\mathcal{T}^Y$ *and* $\mathcal{S}$ *be a subbasis of* $\mathcal{T}^Z$*. If for every subbasic open* $U \in \mathcal{R}_{f(a)}$*, there exists a subbasic open* $V \in \mathcal{S}_{n(a)}$ *such that* $f(n^{-1}(V)) \subset U$*, then* $f$ *is* $n$*-continuous at* $a$*.*

*Proof.* Let $U \in [\mathcal{R}]_{f(a)}$. Since $\mathcal{R}$ is subbasis of $\mathcal{T}^Y$, then there are $k \in \mathbb{N}$ and $U_i \in \mathcal{R}$, with $i = 1, \ldots, k$, such that $U = \bigcap_{i=1}^{k} U_i$. By hypothesis, it follows that for each $U_i \in \mathcal{R}_{f(a)}$, with $i = 1, \ldots, k$, there exists a subbasic open subset $V_i \in \mathcal{S}_{n(a)}$ such that

$f(n^{-1}(V_i)) \subset U_i$. Considering $V = \bigcap\limits_{i=1}^{k} V_i$, then $V \in [\mathcal{S}]_{n(a)}$ and

$$f(n^{-1}(V)) \subset f(n^{-1}(\bigcap_{i=1}^{k} V_i)) \subset \bigcap_{i=1}^{k} f(n^{-1}(V_i)) \subset \bigcap_{i=1}^{k} U_i = U\,.$$

Therefore, it follows from Proposition 2 that $f$ is $n$-continuous in $a$. □

**Corollary 2.** *Let $X$ be a non-empty set, $a \in X$, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions, $\mathcal{R}$ be a subbasis of $\mathcal{T}^Y$ and $\mathcal{B}$ be a basis of $\mathcal{T}^Z$. If for every subbasic open $U \in \mathcal{R}_{f(a)}$, there exists a basic open $V \in \mathcal{B}_{n(a)}$ such that $f(n^{-1}(V)) \subset U$, then $f$ is $n$-continuous at $a$.*

*Proof.* Since every basis of a topology is a subbasis, the corollary follows immediately from the Theorem 4. □

The reciprocal of Theorem 4 is not true. Let $\mathcal{R} = \{(a, b)\colon a, b \in \mathbb{R}\}$ $\mathcal{T}^{\mathbb{R}}$ and $\mathcal{S} = \{(c, \infty)\colon c \in \mathbb{R}\} \cup \{(-\infty, d)\colon d \in \mathbb{R}\}$ be subbasies of $\mathcal{T}^{\mathbb{R}}$. Consider the functions $f\colon (\mathbb{R}, \mathcal{T}^{\mathbb{R}}) \to (\mathbb{R}, \langle\!\langle\mathcal{R}\rangle\!\rangle)$ and $n\colon (\mathbb{R}, \mathcal{T}^{\mathbb{R}}) \to (\mathbb{R}, \langle\!\langle\mathcal{S}\rangle\!\rangle)$ given by $f(x) = n(x) = |x|$. As seen in the Example 1, $f$ is $n$-continuous. However, it is verified that for $x \in \mathbb{R} - \{0\}$ the subbasic open $(0, 2|x|) \in \mathcal{R}_{f(x)}$ is such that $f(n^{-1}(V)) \not\subseteq (0, 2|x|)$, for all subbasic open $V \in \mathcal{S}_{n(x)}$. In fact, the subbasics open around $n(x)$ are of the form $(c, \infty)$, with $c < n(x)$, or $(-\infty, d)$, with $0 < n(x) < d$, and in both cases it follows that

$$f(n^{-1}((c, \infty))) = f((-\infty, -c) \cup (c, \infty)) = \begin{cases} (|c|, \infty) \not\subseteq (0, 2|x|)\,, & \text{if } c \geq 0 \\ [0, \infty) \not\subseteq (0, 2|x|)\,, & \text{if } c < 0 \end{cases}$$

and

$$f(n^{-1}((-\infty, d)) = f((-d, d)) = [0, d] \not\subseteq (0, 2|x|)\,.$$

However, the reciprocal of Corollary 2 is true, as can be seen in the following theorem.

**Theorem 5.** *Let $X$ be a non-empty set, $a \in X$, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions, $\mathcal{R}$ be a subbasis of $\mathcal{T}^Y$ and $\mathcal{B}$ be a basis of $\mathcal{T}^Z$. A function $f$ is $n$-continuous at $a$ if and only if for every subbasic open $U \in \mathcal{R}_{f(a)}$, there exists a basic open $V \in \mathcal{B}_{n(a)}$ such that $f(n^{-1}(V)) \subset U$.*

*Proof.* Let $U \in \mathcal{R}_{f(a)}$. Since $U \in \mathcal{T}^Y_{f(a)}$ and by hypothesis $f$ is $n$- continuous, then there exists $W \in \mathcal{T}^Z_{n(a)}$ such that $f(n^{-1}(W)) \subset U$. Since $W \in \mathcal{T}^Z_{n(a)}$ and $\mathcal{B}$ is a basis of $\mathcal{T}^Z$, then there exists $\mathcal{C} \subset \mathcal{B}$ such that $W = \bigcup\limits_{B \in \mathcal{C}} B$. If $n(a) \in W = \bigcup\limits_{B \in \mathcal{C}} B$, then $n(a) \in B_0$, for some $B_0 \in \mathcal{C}$. Taking $V = B_0$, then

$$f(n^{-1}(V)) \subset f(n^{-1}(W)) \subset U.$$

The reciprocal was proved in the Corollary 2. □

# 4 Generalized compact-open topology

Let $X$ be a non-empty set, $(Y, \mathcal{T}^Y)$, $(Z, \mathcal{T}^Z)$ be topological spaces and consider $n\colon X \to (Z, \mathcal{T}^Z)$ a continuator. The intention is to build a topology to equip the set $\mathsf{C}_n(X,Y)$ of all $n$-continuous functions from $X$ in $(Y, \mathcal{T}^Y)$. For this, let $K$ be a subset of $n(X)$, $U$ be a subset of $Y$ and consider

$$(K,U)_n^Y = \{f \in \mathsf{C}_n(X,Y)\colon f(n^{-1}(K)) \subset U\}, \tag{14}$$

which is simply denoted by $(K,U)_n$, since that $U$ is understood in the context as a subset of $Y$. Consider the family of subsets of $n(X)$ given by

$$\mathcal{K}_n = \{K \in \mathcal{P}(n(X))\colon K \text{ is compact with respect to the topology } \mathcal{T}^{n(X)}\}.$$

Consider also the family of subsets of $\mathsf{C}_n(X,Y)$ given by

$$\mathcal{S}_{n\text{-co}} = \{(K,U)_n \in \mathcal{P}(\mathsf{C}_n(X,Y))\colon K \in \mathcal{K}_n \text{ and } U \in \mathcal{T}^Y\}. \tag{15}$$

**Proposition 5.** *The family $\mathcal{S}_{n\text{-co}}$ given in* (15) *is a generator subbasis of a topology on* $\mathsf{C}_n(X,Y)$.

*Proof.* Note that $\mathsf{C}_n(X,Y) = (\emptyset, Y)_n$ and $(\emptyset, Y)_n \in \mathcal{S}_{n\text{-co}}$, since $\emptyset$ is compact in $n(X)$ and $Y \in \mathcal{T}^Y$. The inclusion $(K,U)_n \subset \mathsf{C}_n(X,Y)$ follows from the equation (14), for all $(K,U)_n \in \mathcal{S}_{n-\text{co}}$. On the other hand, $(\emptyset, Y)_n \subset \bigcup_{(K,U)_n \in \mathcal{S}_{n\text{-co}}} (K,U)_n$. It follows that,

$$\mathsf{C}_n(X,Y) = \bigcup_{(K,U)_n \in \mathcal{S}_{n\text{-co}}} (K,U)_n.$$

Therefore, the family $\mathcal{S}_{n\text{-co}}$ is a generator subbasis of a topology on $\mathsf{C}_n(X,Y)$. □

The topology $\langle\!\langle \mathcal{S}_{n\text{-co}} \rangle\!\rangle$ generated by the generator subbasis $\mathcal{S}_{n\text{-co}}$ is called *$n$-compact-open topology* on $\mathsf{C}_n(X,Y)$ and it is denoted by $\mathcal{T}_{n\text{-co}}$.

Let $(X, \mathcal{T}^X)$ and $(Y, \mathcal{T}^Y)$ be topological spaces and consider a collection given by

$$\mathcal{B} = \{U \times V \in \mathcal{P}(X \times Y) \colon U \in \mathcal{T}^X \text{ and } V \in \mathcal{T}^Y\}.$$

The topology $\langle \mathcal{B} \rangle$ generated by the generator basis $\mathcal{B}$ is called *product topology* on $X \times Y$ (with respect to $\mathcal{T}^X$ and $\mathcal{T}^Y$) and it is denoted by $\mathcal{T}^X \otimes \mathcal{T}^Y$.

Let $(X, \mathcal{T}^X)$ and $(Y, \mathcal{T}^Y)$ be topological spaces and consider $(\mathsf{C}_n(X, Y), \mathcal{T}_{n\text{-co}})$. The function $\varepsilon_n \colon (X \times \mathsf{C}_n(X, Y), \mathcal{T}^X \otimes \mathcal{T}_{n\text{-co}}) \to (Y, \mathcal{T}^Y)$ given by

$$\varepsilon_n(x, f) = f(x)$$

is called the *evaluation function*.

**Theorem 6.** *Let $(X, \mathcal{T}^X)$, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces. Consider $n \colon (X, \mathcal{T}^X) \to (Z, \mathcal{T}^Z)$ a function and $\varepsilon_n \colon (X \times \mathsf{C}_n(X, Y), \mathcal{T}^X \otimes \mathcal{T}_{n\text{-}co}) \to (Y, \mathcal{T}^Y)$ the evaluation function. If $n$ is injective, continuous and $n(X)$ is Hausdorff locally compact, then $\varepsilon_n$ is continuous, that is, $\varepsilon_n \in \mathsf{C}_{id}(X \times \mathsf{C}_n(X, Y), Y)$.*

*Proof.* For this proof the item (ii) of Proposition 2 will be checked. Let $a \in X$, $f \in \mathsf{C}_n(X, Y)$ and $W \in \mathcal{V}^Y_{f(a)}$. Note that $W^\circ \in \mathcal{T}^Y_{f(a)}$. Since $f \in \mathsf{C}_n(X, Y)$, it follows from Theorem 2 that $f \in \mathsf{W}_n(X, Y)$ and it follows from Theorem 3 that

$$n(f^{-1}(W^\circ)) \in \mathcal{T}^{n(X)}_{n(a)}.$$

Since $n(X)$ is Hausdorff locally compact, it follows that (see [8], page 211) there exists a neighborhood $V$ of $n(a)$ such that its closure $\bar{V}$ is compact and

$$n(a) \in V \subset \bar{V} \subset n(f^{-1}(W^\circ)).$$

As $n$ is injective, then $n^{-1}(n(f^{-1}(W^\circ))) = f^{-1}(W^\circ)$ and the subbasic open $(\bar{V}, W^\circ)_n$ contains $f$. In fact,

$$\begin{aligned} \bar{V} \subset n(f^{-1}(W^\circ)) &\Rightarrow n^{-1}(\bar{V}) \subset n^{-1}(n(f^{-1}(W^\circ))) = f^{-1}(W^\circ) \\ &\Rightarrow f(n^{-1}(\bar{V})) \subset f(f^{-1}(W^\circ)) \subset W^\circ \\ &\Rightarrow f \in (\bar{V}, W^\circ)_n \,. \end{aligned}$$

Since $n$ is continuous, then $n^{-1}(V)$ is a neighborhood of $a$. Hence $n^{-1}(V) \times (\bar{V}, W^\circ)_n$ is a neighborhood of $(a, f)$ and

$$\varepsilon_n(n^{-1}(V) \times (\bar{V}, W^\circ)_n) \subset W.$$

In fact, if $(a, f) \in n^{-1}(V) \times (\bar{V}, W^\circ)_n$, then

$$\varepsilon_n(a, f) = f(a) \in f(n^{-1}(V)) \subset f(n^{-1}(\bar{V})) \subset W^\circ \subset W\,.$$

Therefore, $\varepsilon_n \in \mathsf{C}_{id}(X \times \mathsf{C}_n(X, Y), Y)$. □

# 5 Collage of functions with generalized continuity

Let $(Y, \mathcal{T}^Y)$ be a topological space, $X$ be non-empty set, $A$ and $B$ be non-empty subsets of $X$ such that $X = A \cup B$ and $f\colon A \to Y$ and $g\colon B \to Y$ be functions such that $f(x) = g(x)$, for all $x \in A \cap B$. Consider the function $(f * g)\colon X \to (Y, \mathcal{T}^Y)$ given by

$$(f * g)(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}.$$

The function $(f * g)$ is called the *concatenation* of $f$ with $g$ and it is also called *collage function* of $f$ and $g$.

Note that $(f * g)^{-1}(V) = f^{-1}(V) \cup g^{-1}(V)$. In fact, if $x \in (f * g)^{-1}(V)$, that means $(f * g)(x) \in V$. From the definition of $(f * g)$, then $f(x) \in V$ or $g(x) \in V$, hence $x \in f^{-1}(V)$ or $x \in g^{-1}(V)$, consequently, $x \in f^{-1}(V) \cup g^{-1}(V)$. On the other hand, if $x \in f^{-1}(V) \cup g^{-1}(V)$, then $x \in f^{-1}(V)$ or $x \in g^{-1}(V)$, that means that $f(x) \in V$ or $g(x) \in V$ and by the definition of the collage function its follows that $(f * g)(x) \in V$ and therefore $x \in (f * g)^{-1}(V)$.

See an example in which two functions with generalized continuity have concatenation that is not continuous with respect to the concatenation of their continuators. Let $f, p\colon ((-\infty, 0), \mathcal{T}^{(-\infty,0)}) \to (\mathbb{R}, \mathcal{T}^{\mathbb{R}})$ be functions given by $f(x) = x$ and $p(x) = -x$. Also consider $g, q\colon ([0, \infty), \mathcal{T}^{[0,\infty)}) \to (\mathbb{R}, \mathcal{T}^{\mathbb{R}})$ given by $g(x) = q(x) = x$. Note that $f \in \mathsf{C}_p((-\infty, 0), \mathbb{R})$. Consider $x_0 \in (-\infty, 0)$ and $U \in \mathcal{T}^{\mathbb{R}}_{f(x_0)}$. Since $f(x_0) = x_0$, taking $W = U \cap (-\infty, 0)$, then $W \in \mathcal{T}^{(-\infty,0)}_{x_0}$ and $V = p(W) \in \mathcal{T}^{\mathbb{R}}_{p(x_0)}$. Since $p$ is injective, it follows that

$$f(p^{-1}(V)) = f(p^{-1}(p(W))) = f(W) = W \subset U\,.$$

Therefore, $f \in \mathsf{C}_p((-\infty, 0), \mathbb{R})$. As $g = q$, it follows that $g \in \mathsf{C}_q([0, \infty], \mathbb{R})$. Note that $f * g = id$ and $p * q = m$, being $m$ the modulo function. It follows from (3) that the identity function is not $m$-continuous, that is, $(f * g) \notin \mathsf{C}_{(p*q)}(\mathbb{R}, \mathbb{R})$.

The proposition below gives us the condition for the concatenation to have widely continuity with respect to the concatenation of continuators and consequently have concatenation of $n$-continuous functions.

**Theorem 7.** *Let $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $X$, $A$ and $B$ be non-empty sets such that $X = A \cup B$ and $p\colon A \to (Z, \mathcal{T}^Z)$ and $q\colon B \to (Z, \mathcal{T}^Z)$ be functions. Consider $f \in \mathsf{W}_p(A, Y)$, $g \in \mathsf{W}_q(B, Y)$, $f(x) = g(x)$ and $p(x) = q(x)$, for all $x \in A \cap B$. The following statements hold true:*

(i) *If $p(A), q(B) \in \mathcal{T}^Z$, then $(f * g) \in \mathsf{W}_{p*q}(X, Y)$.*

(ii) *If $p$ and $q$ are injective functions and $p(A), q(B) \in \mathcal{F}^Z$, then $(f * g) \in \mathsf{W}_{p*q}(X, Y)$.*

*Proof.* (i) Let $V \in \mathcal{T}^Y$. As $f \in \mathsf{W}_p(A, Y)$ and $g \in \mathsf{W}_q(B, Y)$, it follows from Theorem 3 that

$$p(f^{-1}(V)) \in \mathcal{T}^{p(A)} \quad \text{e} \quad q(g^{-1}(V)) \in \mathcal{T}^{q(B)} \tag{16}$$

Now, notice that

$$\begin{aligned}(p * q)((f * g)^{-1}(V)) &= (p * q)((f^{-1}(V) \cup g^{-1}(V)) \\ &= (p * q)(f^{-1}(V)) \cup (p * q)(g^{-1}(V)) \\ &= p(f^{-1}(V)) \cup q(g^{-1}(V)).\end{aligned}$$

As $(p * q)(X) = p(A) \cup q(B)$, $p(f^{-1}(V)) \in \mathcal{T}^{p(A)}$ e $q(g^{-1}(V)) \in \mathcal{T}^{q(B)}$, it follows that $(p*q)((f*g)^{-1}(V)) \in \mathcal{T}^{(p*q)(X)}$. In fact, as $p(f^{-1}(V)) \in \mathcal{T}^{p(A)}$ and $q(g^{-1}(V)) \in \mathcal{T}^{q(B)}$, then $p(f^{-1}(V)) = W_1 \cap p(A)$ and $q(g^{-1}(V)) = W_2 \cap q(B)$ with $W_1, W_2 \in \mathcal{T}^Z$, this way

$$\begin{aligned}p(f^{-1}(V)) \cup q(g^{-1}(V)) &= (W_1 \cap p(A)) \cup (W_2 \cap q(B)) \\ &= (W_1 \cup W_2) \cap (W_1 \cup q(B)) \cap (p(A) \cup W_2) \cap (p(A) \cup q(B)) \\ &= (W_1 \cup W_2) \cap (W_1 \cup q(B)) \cap (p(A) \cup W_2) \cap (p * q)(X).\end{aligned}$$

As $(W_1 \cup W_2) \cap (W_1 \cup q(B)) \cap (p(A) \cup W_2) \in \mathcal{T}^Z$, then $p(f^{-1}(V)) \cup q(g^{-1}(V)) \in \mathcal{T}^{(p*q)(X)}$. Therefore, $(f * g) \in \mathsf{W}_{p*q}(X, Y)$.

(ii) Let $F \in \mathcal{F}^Y$. As $p$ and $q$ are injective functions and $f \in \mathsf{W}_p(A, Y)$ and $g \in \mathsf{W}_q(B, Y)$, it follows from Proposition 4 that $p(f^{-1}(F)) \in \mathcal{F}^{p(A)}$ and $q(g^{-1}(F)) \in \mathcal{F}^{q(B)}$. Note that

$$(p * q)((f * g)^{-1}(F)) = p(f^{-1}(F)) \cup q(g^{-1}(F))$$

As $(p * q)(X) = p(A) \cup q(B)$, $p(f^{-1}(F)) \in \mathcal{F}^{p(A)}$ e $q(g^{-1}(F)) \in \mathcal{F}^{q(B)}$, it follows that $(p * q)((f * g)^{-1}(F)) \in \mathcal{F}^{(p*q)(X)}$. To verify this, the procedure is similar to the one performed in part (i).
Therefore, $(f * g) \in \mathsf{W}_{p*q}(X, Y)$. □

Let $A, B, C, D$ be non-empty sets, $f\colon A \to C$ and $g\colon B \to D$ be functions. The *cartesian function* between $f$ and $g$ is the function $f \times g\colon A \times B \to C \times D$ given by

$$(f \times g)(x, t) = (f(x), g(t)). \tag{17}$$

**Theorem 8.** *Let $X_1$, $X_2$ be non empty sets, $(Y_1, \mathcal{T}^{Y_1})$, $(Y_2, \mathcal{T}^{Y_2})$, $(Z_1, \mathcal{T}^{Z_1})$, $(Z_2, \mathcal{T}^{Z_2})$ be topological spaces and $n\colon X_1 \to (Z_1, \mathcal{T}^{Z_1})$ and $m\colon X_2 \to (Z_2, \mathcal{T}^{Z_2})$ be functions. If $f \in \mathsf{C}_n(X_1, Y_1)$ and $g \in \mathsf{C}_m(X_2, Y_2)$, then $f \times g \in \mathsf{C}_{n\times m}(X_1 \times X_2, Y_1 \times Y_2)$.*

*Proof.* Let $a = (a_1, a_2) \in X_1 \times X_2$. Consider $U \in \mathcal{B}^{Y_1\times Y_2}_{(f\times g)(a)}$. It follows that $U = U_1 \times U_2$, with $U_1 \in \mathcal{T}^{Y_1}_{f(a_1)}$ and $U_2 \in \mathcal{T}^{Y_2}_{g(a_2)}$. As $f$ é $n$-continuous at $a_1$, for this $U_1 \in \mathcal{T}^{Y_1}_{f(a_1)}$ there exists $V_1 \in \mathcal{T}^{Z_1}_{n(a_1)}$ such that

$$f(n^{-1}(V_1)) \subset U_1. \tag{18}$$

As $g$ is $m$-continuous at $a_2$, for this $U_2 \in \mathcal{T}^{Y_2}_{g(a_2)}$ there exists $V_2 \in \mathcal{T}^{Z_2}_{m(a_2)}$ such that

$$g(m^{-1}(V_2)) \subset U_2. \tag{19}$$

Take $V = V_1 \times V_2 \in \mathcal{B}^{Z_1\times Z_2}_{(n(a_1),m(a_2))}$. This way

$$(n \times m)^{-1}(V) = (n \times m)^{-1}(V_1 \times V_2) = n^{-1}(V_1) \times m^{-1}(V_2)$$

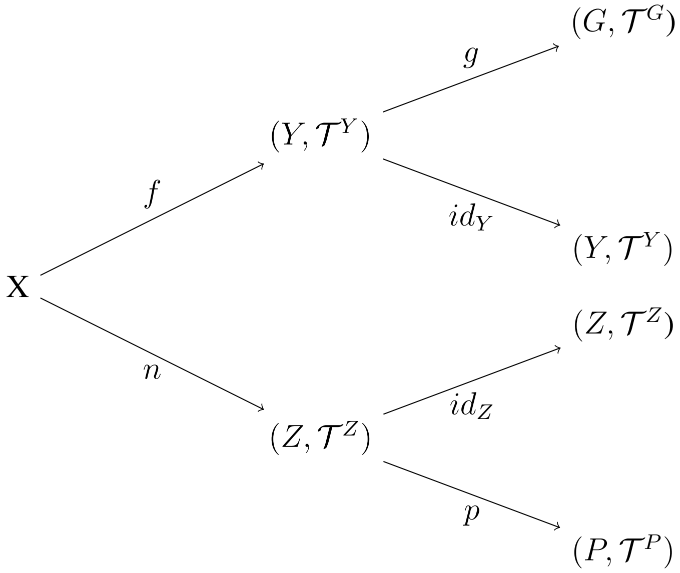
and this implies that

$$\begin{aligned}(f \times g)((n \times m)^{-1}(V_1 \times V_2)) &= (f \times g)(n^{-1}(V_1) \times m^{-1}(V_2)) \\ &= f(n^{-1}(V_1)) \times g(m^{-1}(V_2)) \\ &\subset U_1 \times U_2 \\ &= U\,.\end{aligned}$$

Therefore, $f \times g \in \mathsf{C}_{n\times m}(X_1 \times X_2, Y_1 \times Y_2)$. □

# 6 Composition of functions with generalized continuity

Now we will present results on how compositions of functions with generalized continuity behave. To facilitate the understanding of the Theorem 9 consider the following diagram.




**Theorem 9.** *Let $X$ be non-empty set, $(Y, \mathcal{T}^Y)$, $(Z, \mathcal{T}^Z)$, $(G, \mathcal{T}^G)$ and $(P, \mathcal{T}^P)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$, $g\colon (Y, \mathcal{T}^Y) \to (G, \mathcal{T}^G)$, $n\colon X \to (Z, \mathcal{T}^Z)$ and $p\colon (Z, \mathcal{T}^Z) \to (P, \mathcal{T}^P)$ be functions. The following statements are true:*

(i) *If $f \in \mathsf{F}(X, Y)$ and $g \in \mathsf{C}_{id}(Y, G)$, then $g \circ f \in \mathsf{C}_f(X, G)$*

(ii) *If $f \in \mathsf{C}_n(X, Y)$ and $g \in \mathsf{C}_{id}(Y, G)$, then $g \circ f \in \mathsf{C}_n(X, G)$.*

(iii) *If $n \in \mathsf{F}(X, Z)$ and $id \in \mathsf{C}_p(Z, Z)$, then $n \in \mathsf{C}_{p \circ n}(X, Z)$*

(iv) *If $f \in \mathsf{C}_n(X, Y)$ and $id \in \mathsf{C}_p(Z, Z)$, then $f \in \mathsf{C}_{p \circ n}(X, Y)$*

(v) *If $f \in \mathsf{C}_n(X, Y), g \in \mathsf{C}_{id}(Y, G)$ and $id \in C_p(Z, Z)$, then $g \circ f \in \mathsf{C}_{p \circ n}(X, G)$*

*Proof.* (i) Let $a \in X$ and $U \in \mathcal{T}^G_{(g \circ f)(a)}$. Since $\mathcal{T}^G_{(g \circ f)(a)} = \mathcal{T}^G_{(g(f(a))}$ and by hypothesis $g \in \mathsf{C}_{id}(Y, G)$, then exists $V \in \mathcal{T}^Y_{f(a)}$ such that $g(V) \subset U$. It follows that

$$f(f^{-1}(V)) \subset V \Rightarrow g(f(f^{-1}(V))) \subset g(V) \Rightarrow (g \circ f)(f^{-1}(V)) \subset U\,.$$

Hence, $g \circ f$ é $f$-continuous at $a$. Since $a \in X$ is arbitrary, it follows that $g \circ f \in C_f(X, G)$.

(ii) If $g \in \mathsf{C}_{id}(Y, G)$, it follows from item (i) that $g \circ f \in \mathsf{C}_f(X, G)$. Since $g \circ f \in \mathsf{C}_f(X, G)$ and by hypothesis $f \in \mathsf{C}_n(X, Y)$, it follows from Theorem 1 that $g \circ f \in \mathsf{C}_n(X, G)$.

(iii) Let $a \in X$ and $V \in \mathcal{T}^Z_{n(a)}$. If $id \in \mathsf{C}_p(Z, Z)$, then $id$ is $p$-continuous at $n(a) \in Z$. Since $V \in \mathcal{T}^Z_{n(a)}$ and $id$ is $p$-continuous at $n(a)$, then exists $W \in \mathcal{T}^P_{p(n(a))}$ such that $id(p^{-1}(W)) \subset V$, that is, $p^{-1}(W) \subset V$. It follows that

$$n((p \circ n)^{-1}(W)) = n(n^{-1}(p^{-1}(W))) \subset n(n^{-1}(V)) \subset V\,.$$

Therefore, $n$ é $(p \circ n)$-continuous at $a$. Since $a \in X$ is arbitrary, then $n \in \mathsf{C}_{pon}(X, Z)$.

(iv) By item (iii) above, if $id \in \mathsf{C}_p(Z, Z)$, then $n \in \mathsf{C}_{pon}(X, Z)$. Since $f \in \mathsf{C}_n(X, Y)$ and $n \in \mathsf{C}_{pon}(X, Z)$, it follows from Theorem 1 that $f \in \mathsf{C}_{pon}(X, Y)$.

(iv) If $f \in \mathsf{C}_n(X, Y)$ and $g \in \mathsf{C}_{id}(Y, G)$, it follows from item (ii) that $g \circ f \in \mathsf{C}_n(X, G)$. Since $id \in C_p(Z, Z)$, it follows from item (iii) that $n \in C_{pon}(X, Z)$. If $g \circ f \in \mathsf{C}_n(X, G)$ and $n \in C_{pon}(X, Z)$, it follows from Theorem 1 that $g \circ f \in \mathsf{C}_{pon}(X, G)$. □

**Corollary 3.** *Let $X$ be non-empty set, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $f\colon X \to (Y, \mathcal{T}^Y)$, $g\colon (Y, \mathcal{T}^Y) \to (G, \mathcal{T}^G)$ be functions. If $g \circ f$ is not $f$-continuous, then $g$ is not continuous.*

*Proof.* It is the contrapositive of item (i) of Theorem 9. □

**Proposition 6.** *Let $(X, \mathcal{T}^X)$, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces, $n\colon (X, \mathcal{T}^X) \to (Z, \mathcal{T}^Z)$ be a function and $id_X\colon (X, \mathcal{T}^X) \to (X, \mathcal{T}^X)$ be the identity function. If $n$ is an open function and a injective function, then $id_X \in \mathsf{C}_n(X, X)$.*

*Proof.* Let $a \in X$ and $U \in \mathcal{T}^X_a$. Take $V = n(U)$. Since $n$ is an open function, then $n(U) \in \mathcal{T}^Z_{n(a)}$ and since $n$ is a injective function, then

$$id_X(n^{-1}(V)) = id_X(n^{-1}(n(U))) = id_X(U) = U \subset U.$$

Therefore, $id_X \in \mathbf{C}_n(X, X)$. □

**Corollary 4.** *Let $X$ be non empty set, let $(G, \mathcal{T}^G)$ and $(P, \mathcal{T}^P)$ topological spaces, let $n\colon X \to (Z, \mathcal{T}^Z)$ and $p\colon (Z, \mathcal{T}^Z) \to (P, \mathcal{T}^P)$ functions. If $n$ not is $p \circ n$-continuous, then $p$ not an open function or $p$ not an injective function*

*Proof.* It follows from the contrapositive of item (iii) of Theorem 9, followed by the contrapositive of Proposition 6. □

Let $X$ and $T$ be a non-empty sets and $(Z, \mathcal{T}^Z)$ be a topological space. Consider the *projection function* on the first coordinate $\pi_1\colon X \times T \to X$ and a continuator $n\colon X \to (Z, \mathcal{T}^Z)$. Then the function $n \circ \pi_1\colon X \times T \to (Z, \mathcal{T}^Z)$ is given by

$$(n \circ \pi_1)(x, t) = n(\pi_1(x, t)) = n(x)\,.$$

Finally, the next result relates the generalized continuity of a continuant $f$ relative to a continuator described as a composition of functions with the generalized continuity of the same continuant $f$ relative to a continuator described as a Cartesian function.

**Theorem 10.** *Let $X$ be a non-empty set and $(T, \mathcal{T}^T)$, $(Y, \mathcal{T}^Y)$ and $(Z, \mathcal{T}^Z)$ be topological spaces. Let $f\colon X \times T \to (Y, \mathcal{T}^Y)$, $m\colon T \to (T, \mathcal{T}^T)$ and $n\colon X \to (Z, \mathcal{T}^Z)$ be functions. If $f \in \mathsf{C}_{n\circ\pi_1}(X \times T, Y)$ and $m$ is a surjective function, then $f \in \mathsf{C}_{n\times m}(X \times T, Y)$.*

*Proof.* Let $(a, s) \in X \times T$ and $U \in \mathcal{T}^Y_{f(a,s)}$. Since $f$ is $(n \circ \pi_1)$-continuous, then exists $V \in \mathcal{T}^Z_{(n\circ\pi_1)(a,s)}$ such that $f((n\circ\pi_1)^{-1}(V)) \subset U$. Consider $W = V \times m(T)$ and note that $W \in (\mathcal{T}^Z \otimes \mathcal{T}^T)_{(n(a),m(s))}$ and $W = V \times T$, because $m$ is a surjective function. Since

$$(n \circ \pi_1)^{-1}(V) = \{(x, t) \in X \times T :\ n(x) \in V\} = n^{-1}(V) \times T\,,$$

it follows that

$$\begin{aligned}(n \times m)^{-1}(W) &= \{(x, t) \in X \times T, :\ (n(x), m(t)) \in W\}\\ &= \{(x, t) \in X \times T :\ n(x) \in V \text{ and } m(t) \in T\}\\ &= \{(x, t) \in X \times T :\ x \in n^{-1}(V) \text{ and } t \in m^{-1}(T)\}\\ &= n^{-1}(V) \times m^{-1}(T)\\ &= n^{-1}(V) \times T\\ &= (n \circ \pi_1)^{-1}(V)\,.\end{aligned}$$

Then $f((n\times m)^{-1}(W)) = f((n\circ\pi_1)^{-1}(V)) \subset U$. Therefore, $f \in \mathsf{C}_{n\times m}(X\times T, Y)$. □

# 7 Conclusion

Through the additional considerations about generalized continuity present in this article, it is possible to notice that characterizations and propositions about generalized continuity behave very similar to the usual continuity.

However, there are restrictions to make a composition between two continuants maintaining generalized continuity relative to the composition of their respective continuants.

There are also restrictions for collage of functions with generalized continuity. More specifically, if additional conditions are not required for the continuators, then the collage of two continuants may not have generalized continuity relative to the collage of their respective continuators.

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