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Fantastic and associative filters in pseudo quasi-ordered residuated systems

Filtros fantásticos e associativos em sistemas residuados pseudo quase-ordenados

Daniel A. Romano
International Mathematical Virtual Institute [1]
daniel.a.romano@hotmail.com
ORCID: 0000-0003-1148-3258

Abstract. An interesting generalization of hoop-algebras and commutative residuated lattices is the concept of quasi-ordered residuated systems (shortly QRS) introduced in 2018 by Bonzio and Chajda. Quasi-ordered residuated system is an integral commutative monoid with two internal binary operations interconnected by a residuation connection. This specificity is the reason for the complexity of this algebraic structure and the existence of a significant number of substructures in it, such as various types of filters. The notion of pseudo quasi-ordered residuated systems was introduced and developed in 2022 by this author, omitting the commutativity requirement in QRSs, discussing, additionally, filters in it. Concept of pseudo QRSs is a generalization of the notion of QRSs. In this report, as a continuation of previous research, in addition to the introduction of concepts of fantastic and associative filters in a pseudo quasi-ordered residuated system, their mutual connection between them is discussed, and some examples are presented.

Keywords. Quasi-ordered residuated system (QRS). Pseudo-QRS. Filters in pseudo-QRS. Fantastic filter in pseudo QRS. Associative filter in pseudo QRS.

Resumo. Uma generalização interessante das álgebras de hoops e reticulados residuados comutativos é o conceito de sistemas residuados quase-ordenados (abreviado como QRS), introduzido em 2018 por Bonzio e Chajda. Um sistema residuado quase-ordenado é um monoide comutativo integral com duas operações binárias internas interconectadas por uma

[1] Kordunaška Street 6, 78000 Banja Luka, Bosnia and Herzegovina

conexão de residuação. Essa especificidade é a razão para a complexidade dessa estrutura algébrica e a existência de um número significativo de subestruturas nela, como vários tipos de filtros. A noção de sistemas residuados pseudo quase-ordenados foi introduzida e desenvolvida em 2022 por este autor, omitindo o requisito de comutatividade nos QRSs, e discutindo, adicionalmente, filtros dentro deles. O conceito de pseudo QRSs é uma generalização da noção de QRSs. Neste artigo, como uma continuação de pesquisas anteriores, além da introdução dos conceitos de filtros fantásticos e associativos em um sistema residuado pseudo quase-ordenado, são discutidas suas conexões mútuas e são apresentados alguns exemplos.

Palavras-chave. Sistema residuado quase-ordenado (QRS). Pseudo-QRS. Filtros em pseudo-QRS. Filtro fantástico em pseudo-QRS. Filtro associativo em pseudo-QRS.

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1 Introduction

Non-classical logic has become a formal and useful tool for research in both mathematics and computer science to deal with uncertain information and fuzzy information. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices ([11]). Thus, it is very important to investigate properties of algebraic structures with residuation. By a commutative residuated lattice ([9, 21]), we mean an ordered algebraic structure of the form $(L, \cdot, \wedge, \vee, \rightarrow, 1)$, where $(L, \cdot, 1)$ is a commutative monoid, $(L, \wedge, \vee)$ is a lattice, and the operation '$\rightarrow$' serves as the residuum for the monoid multiplication under the lattice ordering. It is a generalization of ideal lattices of rings. Hoops are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [4, 5]. The concept of filters is an important phenomenon in the mentioned algebraic structures. Different types of filters in mentioned algebraic structures (such as implicative, comparative associative, normal and fantastic filters, for example) have been studied by several researchers. Quasi-ordered residuated system (QRS, for short) is a commutative residuated integral monoids ordered under a quasi-order, introduced by S. Bonzio and I. Chajda in [2] as a generalization of both hoop-algebras and commutative redisuated lattices. In the last few years, the theory of quasi-ordered residuated systems was enriched with more results adding filter on them (see [15, 17, 18, 19]). The determination of the filter in a QRS is somewhat different from the determination of the filter in the previously mentioned algebraic structures.

Non-commutative residuated lattices, called sometimes pseudo-residuated lattices, bi-residuated lattices or generalized residuated lattices. Complete studies on residuated lattices were developed by H. Ono, T. Kowalski, P. Jipsen and C. Tsinakis (see, for example [10, 11, 12]). Pseudo-hoop algebras were presented as non-commutative generalizations of hoop algebras by Georgescu, Leustean and Preoteasa in [8], following after the notions of pseudo-MV algebras in [7] and pseudo-BL algebras ([6]). The pseudo KU-algebras and the pseudo UP-algebras were studied in [13, 14].

The notion of pseudo quaso-ordered residuated system was introduced and analyzed in [20]. In addition, the filters in the pseudo-algebras thus designed were discussed with special reference to the implicative and comparative filters therein.

In this paper, as a continuation of research [20], in addition to the introduction of concepts of fantastic and associative filters in a pseudo quasi-ordered residuated system, some of the important features of such filters and their mutual relations are recognized. The paper is designed in the following way: In Section 2, which comes after the Introductory section, material is presented that enables the potential reader to see, understand and accept the algebraic sub-constructions discussed here without much difficulty. Section 3 is the central part of this article. It gives definitions of fantastic and associative filters in a pseudo QRS and discusses some of their important properties as well as some of their connections to other types of filters.

2 Preliminaries

In this section, the necessary notions and notations, and some of their interrelationships, mostly taken from paper [2, 15, 16, 17, 18, 20], are listed in the order to enable a reader to comfortably follow the presentation in this report. It should be pointed out here that the notations for logical conjunction, logical implication and others have a literal meaning. As usual in mathematical logic, the notation $=:$ in the formula $A =: B$ serves to indicate that A in it is the abbreviation for the formula B.

2.1 Quasi-ordered residuated systems

A relation $\preccurlyeq$ on a non-empty set A is said to be quasi-order on A if it is a reflexive and transitive relation.

In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 1 ([2], Definition 2.1)**.** *A residuated relational system is a structure* $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, R\rangle$*, where* $\langle A, \cdot, \rightarrow, 1\rangle$ *is an algebra of type* $\langle 2, 2, 0\rangle$*, which means that '*$\cdot$*' and*

'$\rightarrow$' are total binary operations on A, and R is a binary relation on A, and satisfying the following properties:

(1) $(A, \cdot, 1)$ *is a commutative monoid,*

(2) $(\forall x \in A)((x, 1) \in R)$,

(3) $(\forall\, x, y \in A)((x \cdot y, z) \in R \iff (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation. A quasi-ordered residuated system is a residuated relational system $\langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order on A.

Exemple 1. *For a commutative monoid A, let $\mathfrak{P}(A)$ be denote the powerset of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A. Then $\langle \mathfrak{P}(A), \cdot, \rightarrow, A, \subseteq\rangle$ is a quasi-ordered residuated system in which the residuum are given by*

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X := \{z \in A : Yz \subseteq X\}).$$

$\square$

Exemple 2. *Any commutative residuated lattice $\langle A, \cdot, \rightarrow, 0, 1, \sqcap, \sqcup, R\rangle$ where R is a lattice quasi-order is a quasi-ordered residuated system.* $\square$

Exemple 3. *Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:*

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

and

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	b	b	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1, 1), (a, 1), (b, 1), (c, 1), (a, a), (b, b), (c, c), (a, b), (a, c)\}.$$

Direct verification it can prove that $\mathfrak{A}$ is a quasi-ordered residuated system. $\square$

Exemple 4. *Let $A = \langle -\infty, 1] \subset \mathbb{R}$ (the real numbers field). If we define '$\cdot$' and '$\rightarrow$' as follows, $(\forall\, u, v \in A)(u \cdot v = min\{u, v\})$ and*

$$u \rightarrow v = 1 \text{ if } u \leqslant v \text{ and } u \rightarrow v = v \text{ if } v < u \text{ for all } u, v \in A,$$

then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, <\rangle$ is a quasi-ordered residuated system. $\square$

The following proposition shows the basic properties of QRSs.

Proposition 1 ([2], Proposition 3.1)**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*

(4) *The operation '$\cdot$' preserves the pre-order in both positions;*

$$(\forall u, v, z \in A)(u \preccurlyeq v \implies (u \cdot z \preccurlyeq v \cdot z \wedge z \cdot u \preccurlyeq z \cdot v));$$

(5) $(\forall u, v, z \in A)(u \preccurlyeq v \implies (v \to z \preccurlyeq u \to z \wedge z \to u \preccurlyeq z \to v))$;
(6) $(\forall u, v, z \in A)(u \cdot (v \to z) \preccurlyeq v \to u \cdot z)$;
(7) $(\forall u, v, z \in A)(u \cdot v \to z \preccurlyeq u \to (v \to z))$;
(8) $(\forall u, v, z \in A)(u \to (v \to z) \preccurlyeq u \cdot v \to z)$;
(9) $(\forall u, v, z \in A)(u \to (v \to z) \preccurlyeq v \to (u \to z))$;
(10) $(\forall u, v, z \in A)((u \to v) \cdot (v \to z) \preccurlyeq u \to z)$;
(11) $(\forall u, v \in A)((u \cdot v \preccurlyeq u) \wedge (u \cdot v \preccurlyeq v))$;
(12) $(\forall u, v, z \in A)(u \to v \preccurlyeq (v \to z) \to (u \to z))$;
(13) $(\forall u, v, z \in A)(v \to z \preccurlyeq (u \to v) \to (u \to z))$.

It is common knowledge that a quasi-order relation $\preccurlyeq$ on a set A generates an equality relation $\equiv_{\preccurlyeq} =: \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equivalence is compatible with the operations in A. Thus, $\equiv_{\preccurlyeq}$ is a congruence on A.

In the light of the previous note, it is easy to see that the following applies:

(7) and (8) give:

(14) $(\forall u, v, z \in A)(u \cdot v \to z \equiv_{\preccurlyeq} u \to (v \to z))$.

Due to the universality of formula (9) (or, due to the commutativity of the multiplication from (14)), we have:

(15) $(\forall u, v, z \in A)(u \to (v \to z) \equiv_{\preccurlyeq} v \to (u \to z))$.

Also, from (11) and (2), it follows

(16) $(\forall u \in A)(u \to u \equiv_{\preccurlyeq} 1)$

In the general case,

(17) $(\forall u, v \in A)(u \preccurlyeq v \iff u \to v \equiv_{\preccurlyeq} 1)$

is valid, which is obtained by referring to (11) and (2).

From the previous analysis it can be concluded that a quasi-ordered residuated system is a generalization of a hoop-algebra (in the sense of [3]) because the following formula

$$(\forall x, y \in A)(x \cdot (x \to y) \equiv_{\preccurlyeq} y \cdot (y \to x))$$

does not have to be a valid formula in the observed algebraic structure in the general case.

Definition 2 ([15], Definition 3.1)**.** *For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a* filter *of $\mathfrak{A}$ if it satisfies conditions*

(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, *and*
(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$.

Let us note that the empty subset of A satisfies the conditions (F2) and (F3). Therefore, $\emptyset$ is a filter in $\mathfrak{A}$. It is shown ([15], Proposition 3.4 and Proposition 3.2), that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the following conditions

(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.

Also, it can be seen without difficulty that $((F3) \wedge F \neq \emptyset) \implies (F2)$ is valid. Indeed, if (F3) holds, then the formula $u \in F \wedge u \preccurlyeq v$, can be transformed into the formula $u \in F \wedge u \rightarrow v \equiv_{\preccurlyeq} 1 \in F$ by (F0) so from here, according to (F3) it can be demonstrate the validity of implications (F2). However, the reverse does not have to be valid.

Exemple 5. *Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:*

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	a	a	1

and

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	b	1	b
c	1	b	1	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (a,a), (a,b), (a,c), (b,b), (c,b), (c,c)\}.$$

Subset $\{1\}$ is a filter of $\mathfrak{A}$. □

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice ([15], Theorem 3.1).

Exemple 6. *Let $\mathfrak{A}$ be as in Example 4. All filters in $\mathfrak{A}$ are in the form of $\langle x, 1]$, for $x \in \langle -\infty, 1]$.* □

2.2 Pseudo quasi-ordered residuated systems

Pseudo quasi-ordered residuated system is a non-commutative generalization of the concept of quasi-ordered residuated systems.

Definition 3 ([20], definition 2.1)**.** *A* pseudo quasi-ordered relational system *is a structure* $\mathfrak{pA} =: \langle A, \cdot, \rightarrow, \rightsquigarrow, 1, \preccurlyeq \rangle$*, where* $\langle A, \cdot, \rightarrow, \rightsquigarrow, 1 \rangle$ *is an algebra of type* $\langle 2, 2, 2, 0 \rangle$ *and* $\preccurlyeq$ *is a quasi-order on* A *satisfying the following properties:*

(1) $(A, \cdot, 1)$ *is a monoid;*

(2) $(\forall u \in A)(u \preccurlyeq 1)$*;*

(3L) $(\forall u, v, z \in A)(u \cdot v \preccurlyeq z \iff u \preccurlyeq v \rightarrow z)$*;*

(3R) $(\forall u, v, z \in A)(u \cdot v \preccurlyeq z \iff v \preccurlyeq u \rightsquigarrow z)$.

We will refer to the operation $\cdot$ *as (non-commutative) multiplication, to* $\rightarrow$ *as its left residuum and to* $\rightsquigarrow$ *as right residuum.*

This system of axioms is hereinafter referred to as pQRS.

Let $\mathfrak{pA} = \langle A, \cdot, \rightarrow, \rightsquigarrow, 1 \rangle$ be a pseudo quasi-ordered residuated system. Then the operation $\cdot$ in $A/\equiv_{\preccurlyeq}$ is commutative if and only if $\rightarrow \equiv \rightsquigarrow$. Indeed, for arbitrary elements $x, y, z \in A$ the following extended equivalence

$$x \cdot y \preccurlyeq z \iff x \preccurlyeq y \rightarrow z \iff x \preccurlyeq y \rightsquigarrow z \iff y \cdot x \preccurlyeq z$$

is valid. In particular, for $z \equiv y \cdot x$ (or for $z \equiv x \cdot y$) we get $x \cdot y \preccurlyeq y \cdot x \preccurlyeq x \cdot y$, i.e. we get $x \cdot y \equiv_{\preccurlyeq} y \cdot x$. Conversely, for arbitrary elements $x, y, z \in A$ the following extended equivalence

$$x \preccurlyeq y \rightarrow z \iff x \cdot y \preccurlyeq z \iff y \cdot x \preccurlyeq z \iff x \preccurlyeq y \rightsquigarrow z$$

is valid. From here, for $x \equiv y \rightarrow z$, or for $x \equiv y \rightsquigarrow z$, we get $x \rightarrow y \preccurlyeq x \rightsquigarrow y \preccurlyeq x \rightarrow y$. In this case, $\mathfrak{pA}/\equiv_{\preccurlyeq}$ is a quasi-ordered residuated system. Thus, we can conclude that the pseudo quasi-ordered residuated system is a generalization of a quasi-ordered residuated system.

Exemple 7. *For a (non-commutative) monoid* A*, let* $\mathfrak{P}(A)$ *be denote the powerset of* A *ordered by set inclusion and '*$\cdot$*' the usual multiplication of subsets of* A*:*

$$(\forall X, Y \in \mathfrak{P}(A)))(X \cdot Y =: \{xy : x \in X \ \wedge \ y \in Y\}).$$

Then $\langle \mathfrak{P}(A), \cdot, \rightarrow, \rightsquigarrow, A, \subseteq \rangle$ *is a quasi-ordered residuated system in which the residuums are given by*

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X := \{z \in A : Yz \subseteq X\}) \text{ and}$$

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightsquigarrow X := \{z \in A : zY \subseteq X\}).$$

□

Exemple 8. *Every pseudo-hop-algebra in the sense of* [8] *is a pseudo quasi-ordered residual system.* □

Exemple 9. *Let* $G = (G, +, -, 0, \vee, \wedge)$ *be an arbitrary lattice ordered group. For an arbitrary element* $u \in G$ *such that* $u \geqslant 0$ *define on the set* $G[u] = [0, u]$ *the following operation*

$$x \cdot y = (x - u + y) \wedge 0,\ x \to y = (y - x + u) \wedge u \text{ and } x \rightsquigarrow y = (u - x + y) \wedge u.$$

Then $\langle G[u], \cdot, \to, \rightsquigarrow, u\rangle$ *is a pseudo quasi-ordered residuated system.* □

Exemple 10. *Let* G *be an arbitrary lattice ordered group and* $nG = \{g \in G : g \leqslant 0\}$*. On* nG *we define the following operations*

$$x \cdot y = x + y,\ x \to y = (y - x) \wedge 0,\ x \rightsquigarrow y = (-x = y) \wedge 0.$$

Then $\langle nG, \cdot, \to, \rightsquigarrow, 0\rangle$ *is a pseudo quasi-ordered residuated system.* □

In the following theorem, we collect and reformulate some results proved in [2].

Theorem 1 ([20], Theorem 2.1)**.** *Let* $\mathfrak{p}\mathfrak{A} =: \langle A, \cdot, \to, \rightsquigarrow, 1, \preccurlyeq\rangle$ *be a pseudo quasi-ordered residuated system. Then:*

(a) $(\forall u, v \in A)((u \preccurlyeq v \iff u \to v \equiv_{\preccurlyeq} 1) \wedge (u \preccurlyeq v \iff u \rightsquigarrow v \equiv_{\preccurlyeq} 1))$
(b) $(\forall u \in A)((u \to u \equiv_{\preccurlyeq} 1) \wedge (u \rightsquigarrow u \equiv_{\preccurlyeq} 1))$
(cL) $(\forall u, v, z \in A)(u \cdot v \to z \equiv_{\preccurlyeq} u \to (v \to z))$
(cR) $(\forall u, v, z \in A)(u \cdot v \rightsquigarrow z \equiv_{\preccurlyeq} v \rightsquigarrow (u \rightsquigarrow z))$
(d) $(\forall u, v, z \in A)(u \preccurlyeq v \implies ((u \cdot z \preccurlyeq v \cdot z) \wedge (z \cdot u \preccurlyeq z \cdot v)))$
(e) $(\forall u, v, z \in A)(u \preccurlyeq v \implies ((z \to u \preccurlyeq z \to v) \wedge (v \to z \preccurlyeq u \to z)))$
(f) $(\forall u, v, z \in A)(u \preccurlyeq v \implies ((z \rightsquigarrow u \preccurlyeq z \rightsquigarrow v) \wedge (v \rightsquigarrow z \preccurlyeq u \rightsquigarrow z)))$
(g) $(\forall u, v \in A)(u \cdot v \preccurlyeq u \ \wedge \ u \cdot v \preccurlyeq v)$
(h) $(\forall u, v \in A)(u \preccurlyeq v \to u \wedge u \preccurlyeq v \rightsquigarrow u)$
(i) $(\forall u \in A)(u \equiv_{\preccurlyeq} 1 \to u \wedge u \equiv_{\preccurlyeq} 1 \rightsquigarrow u)$
(j) $(\forall u, v, z \in A)((v \to z) \cdot (u \to v) \preccurlyeq u \to z)$
(k) $(\forall u, v, z \in A)((u \rightsquigarrow v) \cdot (v \rightsquigarrow z) \preccurlyeq u \rightsquigarrow z)$.

Having regard to (3L) and (3R), it follows directly from statements (j) and (k):
$(\forall x, y, z \in A)(x \to y \preccurlyeq (z \to x) \to (z \to y))$,
$(\forall x, y, z \in A)(x \to y \preccurlyeq (y \to z) \rightsquigarrow (x \to z))$,
$(\forall x, y, z \in A)(x \rightsquigarrow y \preccurlyeq (y \rightsquigarrow z) \to (x \rightsquigarrow z))$,
$(\forall x, y, z \in A)(x \rightsquigarrow y \preccurlyeq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y))$.

Filters in pseudo-hoop algebra have been studied in [8] and [1], for example. In this algebraic structure, a filter F is a non-empty sub-semigroup that satisfies the standard condition for filters. In addition, in [8], Proposition 3.1 it is shown that the condition

$$1 \in F \wedge (\forall a, b \in A)((a \in F \wedge a \to b \in F) \implies b \in F)$$

is equivalent to the condition

$$1 \in F \wedge (\forall a, b \in A)((a \in F \wedge a \rightsquigarrow b \in F) \implies b \in F).$$

In pseudo quasi-ordered residuated systems, the concept of filters in them is determined somewhat differently.

Definition 4 ([20], Definition 2.8). *Let $\mathfrak{pA} =: \langle A, \cdot, \to, \rightsquigarrow, 1, \preccurlyeq \rangle$ be a pseudo quasi-ordered residuated system. A subset F of A is a filter of $\mathfrak{pA}$ if it satisfies the following conditions:*

(F2) $(\forall x, y \in A)((x \in F \wedge x \preccurlyeq y) \implies v \in F)$,

(F3L) $(\forall x, y \in A)((x \in F \wedge x \to y \in F) \implies y \in F)$,

(F3R) $(\forall x, y \in A)((x \in F \wedge x \rightsquigarrow y \in F) \implies y \in F)$.

It is easy to conclude that the sets $\emptyset$ and A are filters in $\mathfrak{pA}$. These are trivial filters in $\mathfrak{pA}$. A filter F of $\mathfrak{pA}$ is proper if $F \neq A$. Besides, if F is a non-empty filter in $\mathfrak{pA}$, then $1 \in F$. Indeed, if $F \neq \emptyset$, then there exists an element $x \in F$. Since $x \preccurlyeq 1$, according to (2), we conclude that $1 \in F$ according to (F2). Thus, $(F \neq \emptyset \wedge (F2)) \implies 1 \in F$.

In addition to the above, the following implications are valid $(F3L) \implies (F2)$ and $(F3R) \implies (F2)$ assuming $F \neq \emptyset$. Assume that (F3L) (i.e. (3FR), respectively) is a valid formula and let elements $x, y \in A$ be such that $x \in F \wedge x \preccurlyeq y$. Then $x \in F \wedge x \to y \equiv_{\preccurlyeq} 1 \in F$ and $x \in F \wedge x \rightsquigarrow y \equiv_{\preccurlyeq} 1 \in F$. Hence $y \in F$ according to (3FL) or (3FR) respectively.

Also, for a non-empty filter F in a pseudo quasi-ordered residuated system $\mathfrak{pA}$, conditions (F3L) and (F3R) are equivalent. Indeed. Let the non-empty set F in a pseudo quasi-ordered residuated system $\mathfrak{pA}$, satisfy the condition (3FR). The assumption $F \neq \emptyset$, implies $1 \in F$. Let $u, v \in A$ be arbitrary elements such that $u \to v \in F$. Then $u \to v \preccurlyeq (v \rightsquigarrow z) \to (u \rightsquigarrow z)$. If we take $z = v$, we get $u \to v \preccurlyeq (v \rightsquigarrow v) \to (u \rightsquigarrow v)$. This means $u \to v \preccurlyeq 1 \to (u \rightsquigarrow v)$ by (b) and $u \to v \preccurlyeq u \rightsquigarrow v$, due to Theorem 1(i). Thus $u \rightsquigarrow v \in F$ by (F2). Therefore, $v \in F$ by (F3R). This means that the implication $F \neq \emptyset \wedge (F3R) \implies (F3L)$ is proved. The implication $F \neq \emptyset \wedge (3FL) \implies (3FR)$ can be proved analogously to the previous proof.

Exemple 11. *Let G be as in Example 9. If K is a convex lattice ordered subgroup of of a lattice ordered group G, then the set $F = \{x \in G[u] : u - x\}$ is a filter in $G[u]$.* □

Exemple 12. *Let G be as in Example 10. If K is a convex lattice ordered subgroup of a lattice ordered group G. The $F = K \cap nG$ is a filter in nG.* □

The concepts of implicative filters and comparative filters can be determined in pseudo-quasi-ordered residuated systems by looking at the corresponding concepts in pseudo-hoop algebras as done in [1]:

Definition 5 ([20], Definition 2.9)**.** *Let $\mathfrak{pA}$ be a pseudo quasi-ordered residuated system and F be a non-empty subset of A. F is said to be an* implicative filter *in $\mathfrak{pA}$ if the following holds:*

(F2) $(\forall x, y \in A)((x \in F \wedge x \preccurlyeq y) \implies v \in F)$,

(IF3) $(\forall x, y, z \in A)((x \in F \wedge x \to ((y \to z) \rightsquigarrow y) \in F) \implies y \in F)$ *and*

(IF4) $(\forall x, y, z \in A)((x \in F \wedge x \rightsquigarrow ((y \rightsquigarrow z) \to y) \in F) \implies y \in F)$.

Definition 6 ([20], Definition 2.11)**.** *Let $\mathfrak{pA}$ be a pseudo quasi-ordered residuated system and F be a non-empty subset of A. A subset F of A is called a* comparative filter *of $\mathfrak{pA}$ if the following holds:*

(F2) $(\forall x, y \in A)((x \in F \wedge x \preccurlyeq y) \implies v \in F)$,

(CF3) $(\forall x, y, z \in A)((x \rightsquigarrow y \in F \wedge x \to (y \to z) \in F) \implies x \to z \in F)$,

(CF4) $(\forall x, y, z \in A)((x \to y \in F \wedge x \rightsquigarrow (y \rightsquigarrow z) \in F) \implies x \rightsquigarrow z \in F)$.

3 The main results

In this section, the concepts of fantastic and associative filters of pseudo quasi-perdered residuated systems are introduced and investigate some of their important properties. In addition to the previous one, the inter-relation of these filters is established.

In what follows, we need the following lemmas:

Lemma 1. *Let F be an non-empty subset of a pseudo quasi-ordered residuated system $\mathfrak{pA}$ that satisfies the condition (F2). Then:*

(18L) $(\forall x \in A)(x \in F \iff 1 \to x \in F)$,

(18R) $(\forall x \in A)(x \in F \iff 1 \rightsquigarrow x \in F)$.

Demonstração. Let a nonempty subset F of a pseudo quasi-ordered residuated system $\mathfrak{pA}$ satisfy the condition (F2) and let $x \in F$. Then from $x \preccurlyeq 1 \to x$, which is valid according to Theorem 1(i), it follows $1 \to x \in F$ according to (F2). Conversely, from

$1 \to x \in F$ and $1 \to x \preccurlyeq x$, which is valid according to Theorem 1(i), we get $x \in F$ according to (F2).

The second part of this statement can be proved in the same way as the first part. □

Corollary 1. *Let F be an non-empty subset of a pseudo quasi-ordered residuated system $\mathfrak{pA}$ that satisfies the condition (F2). Then*

$$(\forall y, v \in A)(v \equiv_{\preccurlyeq} 1 \implies (y \in F \iff v \to y \in F)),$$

$$(\forall y, v \in A)(v \equiv_{\preccurlyeq} 1 \implies (y \in F \iff v \rightsquigarrow y \in F)).$$

Demonstração. First, we have $1 \equiv_{\preccurlyeq} v$ implies $1 \to y \equiv_{\preccurlyeq} v \to y$ and $1 \rightsquigarrow y \equiv_{\preccurlyeq} v \rightsquigarrow y$. Second, $y \in F$ is equivalent to $1 \to y \in F$ and $y \in F$ is equivalent to $1 \rightsquigarrow y \in F$ by Lemma 1. Therefore, $y \in F$ is equivalent to $v \to y \in F$ and $y \in F$ is equivalent to $v \rightsquigarrow y \in F$. □

Lemma 2. *Let F be an non-empty subset of a pseudo quasi-ordered residuated system $\mathfrak{pA}$ that satisfies the condition (F2). Then:*

(19L) $(\forall x, y \in A)(x \to y \in F \iff x \to (1 \to y) \in F)$,

(19R) $(\forall x, y \in A)(x \rightsquigarrow y \in F \iff x \rightsquigarrow (1 \rightsquigarrow y) \in F)$.

Demonstração. On the one hand, from (3L) we get $(x \to y)\cdot x \preccurlyeq y \preccurlyeq 1 \to y$ according to Theorem 1(i). Hence $x \to y \preccurlyeq x \to (1 \to y)$ according to (3L). Now, from $x \to y \in F$ it follows $x \to (1 \to y) \in F$ according to (F2). Conversely, from $x \to (1 \to y) \preccurlyeq x \to (1 \to y)$ it follows $(x \to (1 \to y)) \cdot x \preccurlyeq 1 \to y$ according to (3L). Hence $(x \to (1 \to y)) \cdot x \preccurlyeq 1 \to y \preccurlyeq y$ due to Theorem 1(i). From here we get $(x \to (1 \to y)) \cdot x \preccurlyeq y$ and $x \to (1 \to y) \preccurlyeq x \to y$ according to (3L). Now, from this and from $x \to (1 \to y) \in F$ it follows that $x \to y \in F$ by (F2).

The second part of this statement can be proved in the same way as the first part. □

In addition to the previous one, we also show one characteristic of implicative filters.

Theorem 2. *Let F be a filter in a pseudo quasi-ordered residuated system $\mathfrak{pA}$. Then F an implicative filter in $\mathfrak{pA}$ if and only if F satisfies the following conditions:*

(IF5) $(\forall x, y \in A)((x \to y) \rightsquigarrow x \implies x \in F)$ *and*

(IF6) $(\forall x, y \in A)((x \rightsquigarrow y) \to x \implies x \in F)$.

Demonstração. Let F be an implicative filter in $\mathfrak{pA}$.

Let $x, y \in A$ be such that $(x \to y) \rightsquigarrow x \in F$. Then $1 \to ((x \to y) \rightsquigarrow x) \in F$ by (18L). From here we get $x \in F$ according to (IF3) since $1 \in F$.

Let $x, y \in A$ be such that $(x \rightsquigarrow y) \to x \in F$. Then $1 \rightsquigarrow ((x \rightsquigarrow y) \to x) \in F$ by (18R). From here we get $x \in F$ according to (IF4) since $1 \in F$.

Conversely, let the filter F in $\mathfrak{p}\mathfrak{A}$ satisfy the conditions (IF5) and (IF6). Let us prove that F is an implicative filter in $\mathfrak{p}\mathfrak{A}$.

If $x, y, z \in A$ are arbitrary elements such that $x \rightarrow ((y \rightarrow z) \rightsquigarrow y) \in F$ and $x \in F$, then $(y \rightarrow z) \rightsquigarrow y \in F$ according to (F3L). From here, according to (IF5), we get $y \in F$.

If $x, y, z \in A$ are arbitrary elements such that $x \rightsquigarrow ((y \rightsquigarrow z) \rightarrow y) \in F$ and $x \in F$, then $(y \rightsquigarrow z) \rightarrow y \in F$ according to (F3R). From here, according to (IF6), we get $y \in F$. □

3.1 Fantastic filter

Definition 7. *Let F be a non-empty subset of a pseudo quasi-ordered residuated system $\mathfrak{p}\mathfrak{A}$. F is called a fantastic filter in $\mathfrak{p}\mathfrak{A}$ if it satisfies the following properties:*

(F2) $(\forall x, y \in A)((x \in F \wedge x \preccurlyeq y) \Longrightarrow v \in F)$,

(FF3) $(\forall x, y, z \in A)((z \rightarrow (x \rightarrow y) \in F \wedge z \in F) \Longrightarrow ((y \rightarrow x) \rightsquigarrow x) \rightarrow y \in F)$,

(FF4) $(\forall x, y, z \in A)((z \rightsquigarrow (x \rightsquigarrow y) \in F \wedge z \in F) \Longrightarrow ((y \rightsquigarrow x) \rightarrow x) \rightsquigarrow y \in F)$.

Let us note that every fantastic filter satisfies the condition

(F0) $1 \in F$.

Indeed, $1 \in F$ immediately follows from (F2) since $F \neq \emptyset$.

Theorem 3. *Every fantastic filter F in a pseudo quasi-ordered residuated system $\mathfrak{p}\mathfrak{A}$ is a filter in $\mathfrak{p}\mathfrak{A}$.*

Demonstração. Let F be a fantastic filter of $\mathfrak{p}\mathfrak{A}$ and let $x, y \in A$ be such that $x \in F$ and $x \rightarrow y \in F$ and $x \rightsquigarrow y \in F$.

First, according to (19L), we have $x \rightarrow (1 \rightarrow y) \in F$. Second, since $1 \in F$ and F is a fantastic filter, by (FF3), we have

$$(x \rightarrow (1 \rightarrow y) \in F \wedge 1 \in F) \Longrightarrow ((y \rightarrow 1) \rightsquigarrow 1) \rightarrow y \in F.$$

Now, from the right-hand side of the previous implication follows $y \in F$ according to Corollary 1.

The validity of the formula (F3R) can be proved in a similar way. □

Theorem 4. *Let F be a filter of a pseudo quasi-ordered residuated system $\mathfrak{p}\mathfrak{A}$. Then F is a fantastic filter in $\mathfrak{p}\mathfrak{A}$ if and only if the following holds*

(FF5) $(\forall x, y \in A)(y \rightarrow x \in F \Longrightarrow ((y \rightarrow x) \rightsquigarrow x) \rightarrow y \in F)$,

(FF6) $(\forall x, y \in A)(y \rightsquigarrow x \in F \Longrightarrow ((y \rightsquigarrow x) \rightarrow x) \rightsquigarrow y \in F)$.

Demonstração. Let F be a fantastic filter in a pseudo quasi-ordered residuated system $\mathfrak{pA}$. Then F is a filter in $\mathfrak{pA}$ and, in addition, it satisfies the conditions (F2), (FF3) and (FF4).

Let $x, y \in A$ be such that $y \to x \in F$. Then $1 \to (y \to x) \in F$ by (18L). Now, from $1 \in F$ and $1 \to (y \to x) \in F$ follows $((y \rightsquigarrow x) \to x) \rightsquigarrow y \in F$ according to (FF4). This proves the validity of the formula (FF6).

Let $y \rightsquigarrow x \in F$. Then $1 \rightsquigarrow (y \rightsquigarrow x) \in F$ according to (18R). Now, from $1 \in F$ and $1 \rightsquigarrow (y \rightsquigarrow x) \in F$ follows $((y \to x) \rightsquigarrow x) \to y \in F$ according to (FF3). This proves the validity of the formula (FF5).

Conversely, let F be a filter in a pseudo quasi-ordered residuated system $\mathfrak{pA}$ that satisfies the condition (FF5) and (FF6). Let us prove that F is a fantastic filter in $\mathfrak{pA}$.

Let us take $x, y, z \in A$ such that $z \to (y \to x) \in F$ and $z \in F$. From here we get $y \to x \in F$ according to (F3L) because F is a filter in $\mathfrak{pA}$. Then, we have $((x \to y) \rightsquigarrow y) \to x \in F$ by assumption (FF5). This proves the validity of the formula (FF3).

Let us take $x, y, z \in A$ such that $z \rightsquigarrow (y \rightsquigarrow x) \in F$ and $z \in F$. From here we get $y \rightsquigarrow x \in F$ according to (F3R) because F is a filter in $\mathfrak{pA}$. Then, we have $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in F$ by assumption (FF6). This proves the validity of the formula (FF4). □

3.2 Associative filter

Definition 8. *Let F be a non-empty subset of a pseudo quasi-orderd residuated system $\mathfrak{pA}$. F is called an associative filter of $\mathfrak{pA}$ if the following holds:*

(F2) $(\forall x, y \in A)((x \in F \wedge x \preccurlyeq y) \implies v \in F)$,

(AF3) $(\forall x, y, z \in A)((x \to (y \to z) \in F \wedge x \rightsquigarrow y \in F) \implies z \in F)$,

(AF4) $(\forall x, y, z \in A)((x \rightsquigarrow (y \rightsquigarrow z) \in F \wedge x \to y \in F) \implies z \in F)$.

Obviously, every associative filter in a pseudo quasi-ordered residuated system satisfies the condition (F0).

Theorem 5. *Every associative filter F in a pseudo quasi-ordered residuated system $\mathfrak{pA}$ is a filter in $\mathfrak{pA}$.*

Demonstração. Let F be an associative filter in a pseudo quasi-ordered residuated system $\mathfrak{pA}$.Let us take elements $x, y \in F$ such that $x \in F$ and $x \to y \in F$ (and $x \rightsquigarrow y \in F$). Then $1 \to (x \to y) \in F$ and $1 \rightsquigarrow x \in F$ (and $1 \rightsquigarrow (x \rightsquigarrow y) \in F$ and $1 \to x \in F$) according to (18L) and (18R). In both cases, $y \in F$ follows from here according to (AF3) and according to (AF4) respectively. □

Theorem 6. *Every associate filter in a pseudo quasi-ordered residuated system* $\mathfrak{p}\mathfrak{A}$ *is an implicative filter in* $\mathfrak{p}\mathfrak{A}$.

Demonstração. Let F be an associative filter in pA. This means that F satisfies the conditions (F2), (AF3) and (AF4). Let's prove that it also satisfies the conditions (IF3) and (IF4). In fact, we will prove (FF5) and (FF6).

Let $x, y \in A$ be such that $(x \to y) \rightsquigarrow x \in F$. Then $(x \to y) \rightsquigarrow (1 \rightsquigarrow x) \in F$ by (19R). On the other hand, we have that $(x \to y) \to 1 \in F \iff 1 \in F$ holds for every $x, y \in A$. Indeed, from $x \to y \preccurlyeq 1$ follows $1 \preccurlyeq (x \to y) \to 1$ is valid, so the claim follows from (F2) and (F0). Now, from $(x \to y) \rightsquigarrow (1 \rightsquigarrow x) \in F$ and $(x \to y) \to 1 \in F$, according to (AF4), we get $x \in F$. This proves the validity of the formula (IF5).

Let $x, y \in A$ be such that $(x \rightsquigarrow y) \to x \in F$. Then $(x \rightsquigarrow y) \to (1 \to x) \in F$ by (19L). Now, from here and from the valid formula $(x \rightsquigarrow y) \rightsquigarrow 1 \in F$, we get $x \in F$ according to (AF3). This proves the validity of the formula (IF6). □

Theorem 7. *Every associate filter in a pseudo quasi-ordered residuated system* $\mathfrak{p}\mathfrak{A}$ *is a comparative filter in* $\mathfrak{p}\mathfrak{A}$.

Demonstração. Let F be an associative filter in a pseudo quasi-ordered residuated system $\mathfrak{p}\mathfrak{A}$.

Let $x, y, z \in A$ be arbitrary elements such that $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Then $z \in F$ by (AF3). On the other hand, from (g), for $x =: z$ and $y =: x$, we get $z \preccurlyeq x \to z$ according to (3L). Therefore, $x \to z \in F$ according to (F2). This proves the validity of the formula (CF3).

Let $x, y, z \in A$ be arbitrary elements such that $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \to y \in F$. Then $z \in F$ by (AF4). On the other hand, from (g), for $x =: z$ and $y =: x$, we get $z \preccurlyeq x \rightsquigarrow z$ according to (3R). Therefore, $x \rightsquigarrow z \in F$ according to (F2). This proves the validity of the formula (CF4). □

4 Conclusions and the possibility of advanced research

This work is a continuation of the research on pseudo quasi-ordered residuated systems started with [20]. Here, The concepts of fantastic filters and associative filters in a pseudo quasi-ordered residuated system are introduced and analyzed. The obtained results in this report as the results in the previous paper can be a good base for further and deep analysis of the mentioned types of filters in a pseudo quasi-ordered residuated system. Additionally to the previous, these two articles can be a good basis for recognizing other types of filters in this algebraic structure, such as, for example, normal filters.

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